# On the Bimodular Approximation and Equal Temperaments

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## Abstract

The bimodular approximation, which has been known for over 300 years as an accurate means of computing the relative sizes of small (sub-semitone) musical intervals, is shown to provide a remarkably good approximation to the sizes of larger intervals, up to and beyond the size of the octave. If just intervals are approximated by their *bimodular approximants* (rational numbers defined as frequency difference divided by frequency sum) the ratios between those intervals are also rational, and under certain simply stated conditions can be shown to coincide with the integer ratios which equal temperament imposes on the same intervals. This observation provides a simple explanation for the observed accuracy of certain equal divisions of the octave including 12edo, as well as non-octave equal temperaments such as the fifth-based temperaments of Wendy Carlos. Graphical presentations of the theory provide further insights. The errors in the bimodular approximation can be expressed as *bimodular commas*, among which are many commas featuring in established temperament theory.


## Introduction

Just musical intervals are characterised by small-integer ratios between frequencies. Equal temperaments, by contrast, are characterised by small-integer ratios between intervals. Since interval is the logarithm of frequency ratio, it follows that an equal temperament which accurately represents just intervals embodies a set of rational approximations to the logarithms (to some suitable base) of simple rational numbers. This paper explores the connection between these rational logarithmic approximations and those obtained from a long-established rational approximation to the logarithm function – the bimodular approximation.

After establishing notations, conventions and theoretical foundations we introduce bimodular approximants and list some of their properties. We then embark on an exploration of the relationship between bimodular approximants and equal temperaments. This discussion draws on a study of the errors in approximants and the related topic of bimodular commas. A coordinate system providing a natural environment for the study of equal temperaments is then introduced and illustrated with two types of diagram. An account is then given of some more complex approximant-related structures which have been found in equal temperaments.

## Notations and conventions

We shall use the term interval to refer to the logarithm of frequency ratio, so that intervals combine by addition, rather than multiplication. A *just interval* is an interval having a simple rational frequency ratio.

We shall use a variety of notations to represent specific just intervals.

Intervals specified with a numerical frequency ratio are notated with an underscore, which thus serves as a shorthand for a suitable logarithm function: $\underline{7/4}$.

For 5-limit intervals we use the traditional naming system: P = perfect, M = major, m = minor, A = augmented, D = diminished. A subscript indicates the number of degrees of the scale. Thus $M_6$ = major sixth = $\underline{5/3}$. For certain commonly occurring intervals a single-letter shorthand is used:

o = $P_8$, F = $P_5$, f = $P_4$, M = $M_3$, m = $m_3$, T = $M_2^+$, t = $M_2$, s = $m_2$ = $\underline{16/15}$, X = $A_1$= $\underline{25/24}$.

To distinguish intervals differing by one or more syntonic commas, superscripts $^+$ and $^-$ are appended. Thus $m_3^-$ denotes a just minor third reduced by one syntonic comma (a Pythagorean minor third). The undecorated symbol represents the 'classic' form of the interval, for which the exponent of 3 in the frequency ratio (ignoring the sign) is a minimum, or where this rule is ambiguous, the form with the *maximum* exponent of 5 (ignoring the sign). Some instances to note are:

$A_5 = 2M_3 = \underline{25/16}$ $D_5 = 2m_3 = \underline{36/25}$ $M_2 = \underline{10/9}$

$D_4 = P_8 - A_5 = \underline{32/25}$ $A_4 = P_8 - D_5 = \underline{25/18}$ $m_7 = P_8 - M_2 = \underline{9/5}$

The following notations are used for commas and other small intervals: 0 = unison, c = syntonic comma, p = Pythagorean comma, D (=$D_2$) = diesis, d (= $D^-$) = diaschisma, $D^+$ = major diesis, $D_{min}$ (= $X^{--}$) = minimal diesis, $D_{max}$ (= $X^-$) = maximal diesis (porcupine comma), σ = schisma.

The general interval expressed in logarithmic units is represented by the symbol $J$. A dash indicates a tempered interval ($J'$) and a hat indicates an interval expressed in units of the scale step ($\hat{J}$). These symbols may be combined ($\hat{J}'$).

## Frequency ratios and interval measures

The frequency ratio between two tones having frequency $f_1$ and $f_2$ will be denoted by

$$r = \frac{f_2}{f_1} \tag{1}$$

In the case of a just interval $r$ can be expressed as the ratio of two integers, $n$ and $d$:

$$r = \frac{n}{d} \tag{2}$$

The size of the interval $J$ with frequency ratio $r$ is

$$J = log_{r_u}(r) = \frac{ln(r)}{ln(r_u)} \tag{3}$$

where $r_u$ is the frequency ratio for the unit of interval measurement. When the unit is the cent,

$$r_u = 2^{1/1200} \tag{4}$$

For the purpose of this paper we choose a value of $r_u$ which keeps the algebra simple, namely

$$r_u = e^2 = 7.38906... \tag{5}$$

so that

$$J = \tfrac{1}{2} ln\, r \tag{6}$$

$$r = e^{2J} \tag{7}$$

In this system the unit interval is twice the natural logarithmic unit (the neper), and may conveniently be termed the *dineper* (dNp). Its size in cents is

$$\frac{2400}{ln\, 2} = 3462.468... \tag{8}$$

or about three octaves less 1.4 semitones. This number can be used to convert intervals expressed in dNp to cents when required.

## Bimodular approximants

Given an interval $J$ with frequency ratio $r$, we define its *bimodular approximant* $v(r)$ as

$$v(r) = \frac{r-1}{r+1} = \frac{e^{2J}-1}{e^{2J}+1} = tanh\, J \tag{9}$$

It follows that

$$r = \frac{1+v}{1-v} \tag{10}$$

$$J = \tfrac{1}{2} ln\, r = \tfrac{1}{2} ln\left[\frac{1+v}{1-v}\right] = tanh^{-1} v \tag{11}$$

$$= v + \frac{v^3}{3} + \frac{v^5}{5} + \cdots \tag{12}$$

The first term of this series provides the basis for an approximation to $J$ accurate to second order in $v$

$$J \approx v \tag{13}$$

For frequency ratios less than one, $J$ and $v$ are both negative.

The function $v(r)$ is the order (1,1) Padé approximant of the function $J(r) = ½ \ln r$ in the region of $r = 1$, which has the property of matching the function value and its first and second derivatives at this value of $r$.[1] The approximant function is thus accurate to second order in $r - 1$.

A version of this approximation to the logarithm function, sometimes called the *bimodular approximation*, was described in 1701 by Sauveur[2] and later used by Euler and others.[3]

As the ratio of two polynomials, $v(r)$ is a rational function. In the case of just intervals, for which $r$ is a rational number ($n/d$), $v$ is also a rational number, being expressible as

$$v = \frac{n-d}{n+d} = \frac{j}{g} \tag{14}$$

where $j$ and $g$ are integers. $v$ may thus be termed a *rational approximant* of $J$.

Example: The perfect fifth F has frequency ratio $r = 3/2$ and approximant $v = (3 - 2)/(3 + 2) = 1/5 = 0.2$. The accurate size of the interval is $J = ½ \ln(3/2) = 0.20273...$ dNp.

The general approximant has the following relationship to the frequencies $f_1$ and $f_2$:

$$v = \frac{f_2 - f_1}{f_2 + f_1} = \frac{frequency\ difference}{frequency\ sum} = \tfrac{1}{2}\frac{frequency\ difference}{mean\ frequency} \tag{15}$$

This expression can be viewed as an approximation to an integral in which the interval is built up from infinitesimal frequency increments:

$$J = \tfrac{1}{2} ln\left(\frac{f_2}{f_1}\right) = \tfrac{1}{2}\int_{f_1}^{f_2} \frac{df}{f} \tag{16}$$

Applying one-point Gaussian quadrature to this integral gives the approximation

$$J \approx \tfrac{1}{2}\frac{\Delta f}{\bar{f}} = \frac{f_2 - f_1}{f_2 + f_1} = v \tag{17}$$

The bimodular approximation is of historical importance as a means of computing logarithms and the sizes of sub-semitone intervals. The purpose of this paper is to show that it can also be applied usefully to larger intervals, in which role it provides insights into the representation of just intervals in equal temperaments.

A graphical representation of the bimodular approximation is shown in Figure 1. Some common 5-limit just intervals are represented on the diagram by lines radiating from the origin, each line having a gradient equal to the interval's frequency ratio. The error in the approximation only becomes visibly apparent for intervals approaching octave size.

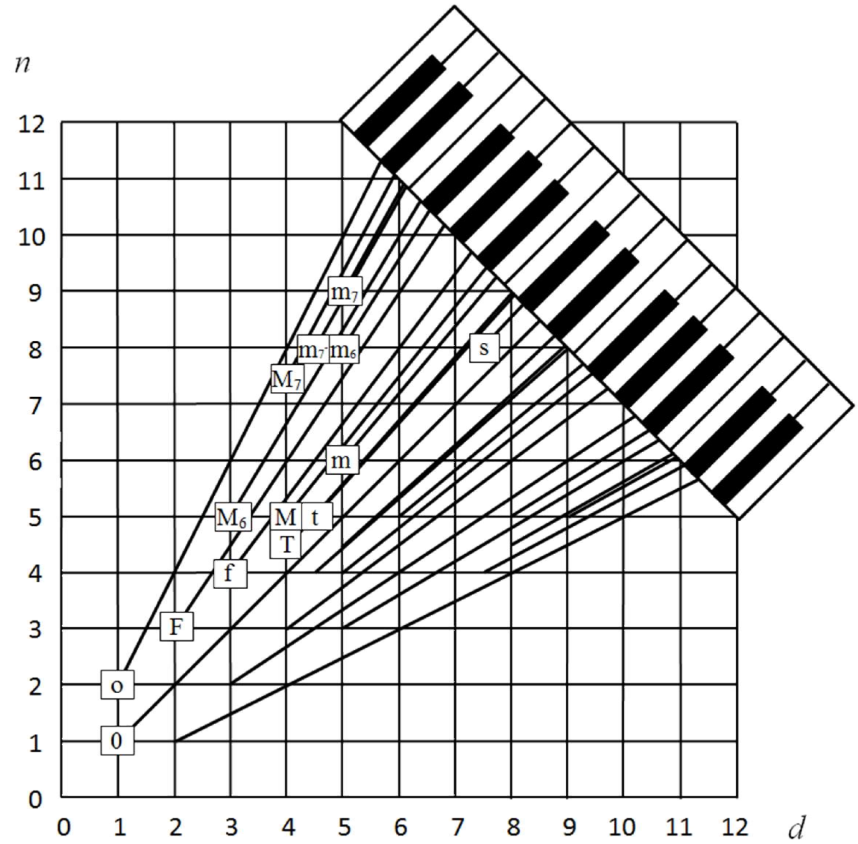


Figure 1. Graphical representation of the bimodular approximation

The ratio of an interval to its approximant proves to be a useful quantity and will be termed the interval's *gauge* ($G$):

$$G = \frac{J}{v} = \frac{\tanh^{-1} v}{v} = 1 + \tfrac{1}{3}v^2 + \tfrac{1}{5}v^4 + \cdots \tag{18}$$

The value of $G$ approaches unity from above as $v$ becomes small. As well as a ratio, $G$ can be considered as an interval. This interval, the result of multiplying $J$ by $1/v$, is somewhat greater than 1dNp (≈ 3462 cents).

For tempered intervals $J'$ we also define a *tempered gauge* $G'$:

$$G' = \frac{J'}{v} \tag{19}$$

Like its untempered counterpart, the tempered gauge approximates 1dNp. When expressed in steps of an equal temperament it will be denoted by $\hat{G}'$.

## Bimodular approximants of common intervals

Approximants for the first 31 superparticular intervals – the intervals between consecutive harmonics of a fundamental frequency – are shown in Figure 2. The harmonics are numbered on the vertical lines and all intervals transposed to lie within the octave, with harmonics of the note C, including the septimal seventh ('B𝄫' in sagittal notation), indicated at the bottom of the figure. These approximants are reciprocals of odd integers. Those in each row of the diagram sum to about ½ ln(2) = 0.34657... (an accurate octave), with increasing accuracy in successive rows.

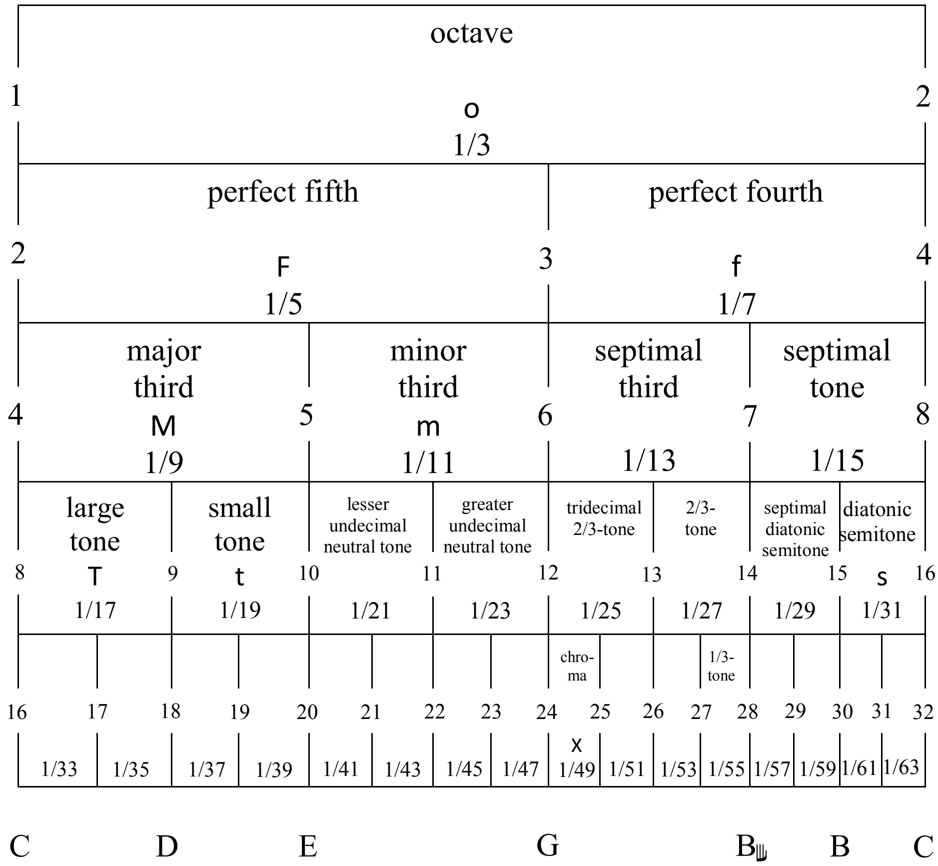


Figure 2. Approximants of low-order superparticular intervals

The approximants of some other common intervals are listed in Table 1.

Inspection of the approximants in Figure 2 and Table 1, and others obtainable quickly by mental calculation, reveals many near-rational relationships between just intervals:

Two perfect fourths ($r = 4/3$, $v = 1/7$) approximate a minor seventh ($r = 9/5$, $v = 2/7$)

Three major thirds ($r = 5/4$, $v = 1/9$) or two 7/5s ($v = 1/6$) or five 8/7s ($v = 1/15$) approximate an octave ($r = 2/1$, $v = 1/3$)

Three 8/7s ($v = 1/15$) or two 11/9s ($v = 1/10$) approximate a perfect fifth ($r = 3/2$, $v = 1/5$)

Two 9/7s ($v = 1/8$) approximate a major sixth ($r = 5/3$, $v = 1/4$)

Three 11/10s ($v = 1/21$) approximate a perfect fourth ($r = 4/3$, $v = 1/7$)

Three small tones ($r = 10/9$, $v = 1/19$) approximate an 11/8 ($v = 3/19$)

Three large tones ($r = 9/8$, $v = 1/17$) approximate a 10/7 ($v = 3/17$)

Seven chromas ($r = 25/24$, $v = 1/49$) approximate a perfect fourth ($r = 4/3$, $v = 1/7$)

These examples demonstrate the value of approximants as a tool for making quick estimates of interval sizes, or explaining seemingly fortuitous relationships between them. It will be shown that the underlying relationships also provide insights into the properties of tuning systems in which the approximate numerical relationships are rendered exact by tempering.

| *Just interval, J* | *Symbol* | *Frequency ratio, r* | *Approximant, v* |
|---|---|---|---|
| Perfect twelfth | $P_{12}$ | 3/1 | 1/2 |
| Major seventh | $M_7$ | 15/8 | 7/23 |
| Greater just minor seventh | $m_7$ | 9/5 | 2/7 |
| Pythagorean minor seventh | $m_7^-$ | 16/9 | 7/25 |
| Major sixth | $M_6$ | 5/3 | 1/4 |
| Minor sixth | $m_6$ | 8/5 | 3/13 |
| Classic augmented fifth | $A_5$, 2M | 25/16 | 9/41 |
| Classic diminished fifth | $D_5$, 2m | 36/25 | 11/61 |
| Septimal augmented fourth | $\underline{10/7}$ | 10/7 | 3/17 |
| Classic augmented fourth | $A_4$ | 25/18 | 7/43 |
| Septimal diminished fifth | $\underline{7/5}$ | 7/5 | 1/6 |
| Septimal major third | $\underline{9/7}$ | 9/7 | 1/8 |
| Semitone maximus | $m_2^+$, $s^+$ | 27/25 | 1/26 |
| Chroma | $A_1$, X | 25/24 | 1/49 |
| Syntonic comma | c | 81/80 | 1/161= 1/(7×23) |
| Diesis | $D_2$, D | 128/125 | 3/253 = 3/(11×23) |
| Major diesis | $D^+$ | 648/625 | 23/1273 = 23/(19×67) |

Table 1. Approximants of some other common intervals

## Relationships among intervals, frequency ratios and approximants

The following results relating to bimodular approximants $v$ of intervals $J$ with frequency ratio $r$ follow directly from the definition (eqn 9).

$$(r+1)(1-v) = 2 \quad (20)$$

$$J_1 + J_2 = 0 \quad \Leftrightarrow \quad r_1 r_2 = 1 \quad \Leftrightarrow \quad v_1 + v_2 = 0 \quad (21)$$

$$J_1 + J_2 + J_3 = 0 \quad \Leftrightarrow \quad r_1 r_2 r_3 = 1 \quad \Leftrightarrow \quad v_1 + v_2 + v_3 + v_1 v_2 v_3 = 0 \quad (22)$$

Equation 21 states that reversing the sign of $J$ reverses the sign of $v$. Eqn 22 (of which eqn 21 is a special case) expresses the summation rule for the tanh function:

$$J_+ = J_1 + J_2 \quad \Leftrightarrow \quad v_+ = \tanh(J_+) = \frac{v_1 + v_2}{1 + v_1 v_2} \quad (23)$$

where $J_+ = -J_3$ and $v_+ = -v_3$. The term $v_1 v_2 v_3$ in eqn 21 can be viewed as the summation error incurred when approximants take the place of accurate intervals.

It is sometimes convenient to work with the wavelength ratio, $w = 1/r$. In this formulation the relationships have exact symmetry:

$$v = \frac{(1-w)}{(1+w)}, \qquad w = \frac{(1-v)}{(1+v)}, \qquad (1+w)(1+v) = 2 \quad (24)$$

and eqns 21 and 22, expressed in terms of $w$, have the following duals:

$$v_1 v_2 = 1 \quad \Leftrightarrow \quad w_1 + w_2 = 0 \quad (25)$$

$$v_1 v_2 v_3 = 1 \quad \Leftrightarrow \quad w_1 + w_2 + w_3 + w_1 w_2 w_3 = 0 \quad (26)$$

Here the values of $v$ and $w$ are unconstrained: $|v| > 1$ and $w < 0$ are permitted. In such cases $J$ is complex, but physical meaning can always be restored by making a substitution using eqn 25. Thus eqn 26 (considered as a general relationship between three intervals) can be expressed alternatively as

$$\frac{v_1 v_2}{v_3} = 1 \quad \Leftrightarrow \quad w_1 + w_2 - w_3 - w_1 w_2 w_3 = 0 \qquad (27)$$

Another option is to avoid calculations involving $J$ and work exclusively with $v$, $r$ and $w$, for which the given relationships may be applied with impunity over the full range of real values.

## Equal temperaments

An equal temperament (ET) is a tuning system constructed from equal interval steps and representing an approximation to just intonation. It is usually defined by dividing some interval (the *base interval*) into an integer number of equal parts, this number being termed the *cardinality* of the ET. The base interval is very commonly the octave, and usually tuned pure, but other choices are possible. Octave-based temperaments are referenced by various shorthand notations such as 12edo, 12ed2, 12et or 12tet, '12' in this case denoting the cardinality, 'ed' = 'equal division', and '(t)et' = '(tone) equal temperament'. 'o' or '2' is an abbreviation of the name or frequency ratio of the base interval (in this case the octave). Some writers prefer to confine the use of 'edo' to tunings for which there is no implied intention to approximate just intonation, but we shall not make this distinction.

While the pitches of an ET are fixed, their musical interpretation can differ depending on context (A-sharp/B-flat) and between different conceptual versions of the temperament. If the aim of the temperament is to produce an approximation to just intonation (JI) two tuning approaches are commonly adopted.

The first is based on a *val*, a vector defining the sizes of tempered intervals representing the prime number frequency ratios (2, 3, 5, etc.) up to some *prime limit*. Other tempered intervals are constructed by combining the tempered primitive intervals in amounts equal to the powers of 2, 3, 5, etc. in their frequency ratios. The *patent val* tunes each primitive interval to the nearest scale step.

The second approach tunes every interval to the nearest scale step. In the case of the more complex intervals this produces tunings which differ from those derived from the patent val – a phenomenon known as *inconsistency*.

#### ***Example: 29edo***

For 29edo the 5-limit patent val is <29 46 67|, but the meantone version of this temperament, '29c', has val <29 46 68|. An augmented fourth (25/16) is tempered to 18 steps by the patent val, to 20 steps by 29c, and to 19 steps by a version of 29edo adopting the nearest-step tuning approach.

## Near-equal progressions of just intervals

A sequence of approximants

$$v = \frac{j}{g} \qquad (j = 0, 1, 2, \ldots) \qquad (28)$$

where $g$ is a fixed integer and $j$ takes a succession of integer values, correspond to intervals with frequency ratios

$$r = \frac{g + j}{g - j} \qquad (j = 0, 1, 2, \ldots) \qquad (29)$$

Since the approximants are all multiples of the constant $1/g$, we can expect that the associated intervals, provided they do not become too large, will lie close to multiples of a unit step, and will thus form an approximation to an equal temperament.

This expectation is confirmed by the following examples:

The approximants (0/9, 1/9, 2/9, 3/9) with frequency ratios (9/9, 10/8, 11/7, 12/6) correspond to intervals (0, 386, 782, 1200) cents, which are close to steps of 3edo.

The approximants (0/1, 1/12, 2/12, 3/12, 4/12) with frequency ratios (12/12, 13/11, 14/10, 15/9, 16/8) correspond to intervals (0, 289, 583, 884, 1200) cents, which are close to steps of 4edo.

The approximants (0/1, 1/15, 2/15, 3/15, 4/15, 5/15) with frequency ratios (15/15, 16/14, 17/13, 18/12, 19/11, 20/10 with) correspond to intervals (0, 231, 464, 702, 946, 1200) cents, which are close to steps of 5edo.

The approximants (0/1, 1/18, 2/18, 3/18, 4/18, 5/18, 6/18) with frequency ratios (18/18, 19/17, 20/16, 21/15, 22/14, 23/13, 24/12) correspond to intervals (0, 193, 386, 583, 782, 988, 1200) cents, which are close to steps of 6edo.

The approximants (0/35, 1/35, 2/35, 3/35, 4/35, 5/35, 6/35, 7/35) correspond to frequency ratios (35/35, 18/17, 37/33, 19/16, 39/31, 4/3, 41/29, 3/2) and intervals (0, 99, 198, 297, 397, 498, 599, 702) cents, which are close to steps of 12edo.

A sequence of this sort is made up of the intervals between pairs of harmonics which share a common sum (or average) frequency. We shall call it a *near-equal progression*.

If two approximants are expressed as fractions with the same denominator they can be seen to be members of such a sequence, and this provides a simple way to picture relationships between intervals whose approximants bear a simple ratio to one another.

For example, our earlier observation that the major third ($r$ = 5/4) and the octave ($r$ = 2/1 = 6/3) are roughly in their approximant ratio 1/9:1/3 = 1:3 can be viewed alternatively in terms of intervals between the inner and outer pairs of harmonics in the following segment of the harmonic series: ($\underline{\underline{3}}$ $\underline{4}$ $\underline{5}$ $\underline{\underline{6}}$). These pairs have equal frequency sums and frequency differences in the ratio 1:3.

## Approximant ratio matching in equal temperaments

By developing the forgoing ideas we shall show that every equal temperament contains readily identifiable sets of tempered intervals whose relative sizes match the ratios of the associated bimodular approximants. Such sets will be described as *approximant-matched*. When a temperament has approximant-matched sets containing intervals of low complexity it often represents a good approximation to just intonation, within a certain scope such as that defined by a prime limit.

For a pair of tempered intervals $J_1'$ and $J_2'$ with approximants $v_1'$ and $v_2'$, approximant-matching means

$$\frac{J_2'}{J_1'} = \frac{v_2}{v_1} \tag{30}$$

which implies that $J_1{}'$ and $J_2{}'$ have the same tempered gauge:

$$G_1' = \frac{J_1'}{v_1} = \frac{J_2'}{v_2} = G_2' \tag{31}$$

or (expressed in steps of the temperament)

$$\hat{G}_1' = \frac{\hat{J}_1'}{v_1} = \frac{\hat{J}_2'}{v_2} = \hat{G}_2' \tag{32}$$

Tempered gauges have values close to 1 dNp (about three octaves). In cases of practical interest they very commonly correspond to an integer number of scale steps.

The following are some examples of approximant-matching in equal temperaments.

***Example: approximant-matching in 12edo***

Attempts to account for the success of 12edo as a 5-limit tuning have ranged from appeals to pure coincidence (which in view of the temperament's extraordinarily accurate rendering of fourths and fifths is less than convincing) to arguments based on continued fractions, which do little more than express the apparent coincidence in a different way. An altogether more satisfactory, and very simple, explanation is provided by bimodular approximants.

If the perfect fifth and the perfect fourth – the most fundamental intervals after the octave – are adjusted so that their ratio matches that of their bimodular approximants, 1/5:1/7 or 7:5, and their sum is tuned to a pure octave, the result is a division of the octave into 7 + 5 = 12 equal parts: 12edo.

An analysis of errors (below) shows that the tuning of approximant matched intervals is particularly accurate when the intervals are of similar size, not too large, and adjusted so that their sum is a pure interval – criteria which are all satisfied in the case of the (f, F) approximant match in 12edo. This provides a simple rationale for the remarkable accuracy of this temperament in the 3-limit, which is characterised by an error in the fifth of less than 2 cents.

The approximate 5:7 ratio between the fourth (4/3) and the fifth (3/2) can be appreciated intuitively by considering the harmonic series segment (14 15 16 17 18 19 20 21) in which intervals of a fourth and a fifth between the pairs of singly and doubly underlined harmonics are associated with equal frequency sums and frequency differences of 5 and 7, respectively. The associated near-equal progression also provides an explanation for the well-known fact that one and three steps of 12edo closely approximate 18/17 and 19/16, respectively.

Approximant ratios also explain 12edo's somewhat less impressive accuracy in the 5-limit and the 7-limit. The approximant-matched set containing the perfect fifth and fourth has a tempered gauge of 35 steps:

$$\hat{G}' = \frac{5}{1/7} = \frac{7}{1/5} = 35 \qquad (33)$$

Also in this set is the 5-limit minor seventh (9/5), as can be seen by noting that in 12edo the ratio of the perfect fourth to the minor seventh (5:10) matches the ratio of the corresponding approximants, 1/7 : 2/7 (a property shared by all temperaments in the meantone family).

Another important approximant-matched set for 12edo comprises the major third, the 7/5 interval, the major sixth and the octave. These matched interval sets can be summarised as follows, with each set followed in brackets by the tempered gauge $\hat{G}'$ which its members share:

$f' : F' : m_7' = 1/7 : 1/5 : 2/7 = 5 : 7 : 10 \qquad (\hat{G}' = 35)$

$M' : 7/5' : M_6' : o' = 1/9 : 1/6 : 1/4 : 1/3 = 4 : 6 : 9 : 12 \qquad (\hat{G}' = 36)$

A more extensive (but not exhaustive) list of approximant-matched 7-limit intervals in 12edo is given below. Here the intervals in each matched set (with dashes omitted) are written in a bracket, followed by the tempered gauge as a subscript.

$(T, 10/7)_{34}$ $(f, F, m_7)_{35}$ $(M, 7/5, M_6, o)_{36}$

$(21/16, 25/12, 27/10)_{37}$ $(t, 14/5, P_{12})_{38}$ $(7/6, 10/3)_{39}$

***Example: approximant-matching in 22edo***

22edo is another temperament providing a workable approximation to the 5-limit. Its matches include the following:

(s, 35/27)$_{62}$ (M, f)$_{63}$ (9/7, $M_6$)$_{64}$ (7/6, F, $m_6$)$_{65}$
(m, 7/5, 7/4, o, 15/7)$_{66}$ (T, $m_{10}$, 25/9)$_{68}$

***Example: approximant-matching in 53-edo***

The matches of 53edo, a very accurate 5-limit temperament, include

(T, M)$_{153}$ (m, f, $A_4^+$)$_{154}$ (s, F)$_{155}$ ($s^+$,$m_6$, $M_6$)$_{156}$

***Example: approximant-matching in fifth-based equal temperaments***

Approximant ratio matching is not confined to octave-based equal temperaments. The *alpha*, *beta* and *gamma* temperaments of Wendy Carlos,[4] which are normalised to a pure perfect fifth, can be understood in terms of approximant-matched pairs of intervals drawn from the set (m, M, F).

The *alpha* scale divides the fifth into 9 equal parts, setting (M, F) in their approximant ratio 1/9 : 1/5 = 5 : 9. The same match is exploited by 31edo.

The *beta* scale divides the fifth into 11 equal parts, setting (m, F) in their approximant ratio 1/11 : 1/5 = 5 : 11. The same match is exploited by 19edo.

The *gamma* scale divides the fifth into 20 equal parts, setting (m, M) in their approximant ratio 1/11 : 1/9 = 9 : 11. The same match is exploited by 34edo.

The *gamma* scale achieves the highest accuracy of these three by approximant-matching a pair of small, similarly sized intervals whose sum is normalised – principles which have already been mentioned in relation to 12edo and which will be discussed further in connection with approximant errors.

*Miracle temperament* divides the fifth into 6 equal parts, exploiting the following approximant ratios (amongst others):

8/7′ : 11/9′ : 7/5′ : F′ = 1/15 : 1/10 : 1/6 : 1/5 = 2 : 3 : 5 : 6 $(\hat{G}' = 30)$

***Example: approximant-matching in 88cET***

Approximant-matching also sheds light on a group of temperaments known by the generic name 88 cent equal temperament (88cET). The member of this temperament family formed by dividing a pure major tenth into 18 equal steps (giving a step of 88.129 cents) approximant-matches the pair of intervals (F, $M_6$), which have a ratio 1/5 to 1/4 or 4 : 5. A fifth of 8 steps and a major sixth of 10 steps sum to an18-step pure major tenth. This is another example of accuracy achieved by normalising the sum of a pair of similarly-sized approximant-matched intervals.

The 11/9 and 9/7 intervals (which also have approximants in the ratio 4 : 5) are other members of this matched set:

11/9′ : 9/7′ : F′ : $M_6$′ = 1/10 : 1/8 : 1/5 : 1/4 = 4 : 5 : 8 : 10 $(\hat{G}' = 40)$

Other properties of 88cET, including its accurate 7/6 and 7/4, can be understood by considering temperaments in this group as slightly retuned versions of every third step of 41edo.

Among the 11-limit approximant-matched sets for 41edo are

(7/6, M, 15/11)$_{117}$ (T, 64/55, f, 10/7)$_{119}$ (8/7, 11/9, 9/7, 7/5, F, $M_6$)$_{120}$
(m, 72/49, 7/4)$_{121}$ (21/20, 27/14, o)$_{123}$ (16/15, 21/10)$_{124}$
(11/10, 25/11, $M_{10}$)$_{126}$ ($s^+$, $P_{12}$, 49/16)$_{130}$

***Example: approximant-matching in Bohlen-Pierce equal temperament***

The Bohlen-Pierce equal temperament is a 13-part equal division of the tritave (perfect twelfth) which provides an accurate tuning for just intervals involving odd-number frequency ratios. It can be understood in terms of the following approximant-matched set of intervals which sum to a 13-step normalised perfect twelfth:

$$\underline{9/7}' : \underline{7/5}' : \underline{5/3}' = 1/8 : 1/6 : 1/4 = 3 : 4 : 6 \qquad (\hat{G}' = 24)$$

This temperament can be considered as a retuned version of every fifth step of 41edo, with which it shares these matches.

## Equal temperaments derived from approximant ratio matches

If an octave-based equal temperament is assessed purely on the basis of its accuracy in the 3-limit, a single approximant match between 3-limit intervals (which, as we shall show, is associated with a single tempered-out comma) is sufficient to define it uniquely (subject to an arbitrary integer multiple, since subdivision preserves ratios). It is thus possible to derive a succession of 3-limit equal temperaments from the requirement that approximant ratios for certain pairs of 3-limit intervals are matched.

In the first example above, 12edo was derived as a 3-limit temperament by approximant-matching the fourth and the fifth. Further examples of this technique are listed in Table 2. In each row a 3-limit equal temperament is derived from approximant ratio matching for a pair of 3-limit source intervals. The procedure yields a dozen well-known temperaments.

| Approximant-matched 3-limit intervals | Approximants | Octave-cardinality of associated equal temperament | Error in fifth (cents) |
|---|---|---|---|
| (F, o) | (1/5, 1/3) | 5 | 18.0 |
| (f, o) | (1/7, 1/3) | 7 | -16.2 |
| (f, F) | (1/7, 1/5) | 12 | -2.0 |
| (T, o) or (F, $m_7^-$) | (1/17,1/3) or (1/5,7/25) | 17 | 3.9 |
| (o, $M_9^+$) | (1/3, 5/13) | 26 | -9.6 |
| (T, F) | (1/17, 1/5) | 29 | 1.5 |
| (T, f) | (1/17, 1/7) | 41 | 0.5 |
| ($M_6^+$,o) | (11/43, 1/3) | 43 | -4.3 |
| ($m_7^-$, o) | (7/24, 1/3) | 50 | -6.0 |
| (F, $M_6^+$) | (1/5, 11/43) | 74 | -4.7 |
| ($M^+$, F) | (17/145, 1/5) | 99 | 1.1 |
| ($m^-$, F) | (5/59, 1/5) | 101 | -1.0 |

Table 2. Low-cardinality 3-limit equal temperaments derived from approximant-matching

This table provides another illustration of the tendency for temperaments to be accurate when their approximant-matched intervals are both small and similar in size.

If this technique is extended to the 5-limit a single approximant ratio match is insufficient to define the temperament, but is consistent with a family of temperaments (including temperaments of rank 2 and higher) which temper out a specific comma. To illustrate the principle we shall use a restricted set of source intervals (T, m, M, f, F) and consider only equal temperaments belonging to the meantone, diaschismic and schismic families (which temper out, respectively, the syntonic comma, the diaschisma and the schisma).

The temperaments generated by this procedure are shown in Table 3, in which minimal octave cardinalities of meantone, diaschismic and schismic derived temperaments are displayed in the body of the table for all combinations of the source intervals (which are labelled with their approximants in brackets).

| | F (1/5) | f (1/7) | M (1/9) | m (1/11) |
|---|---|---|---|---|
| f (1/7) | 12, 12, 12 | | | |
| M (1/9) | 31, 46, 77 | 43, 22, 65 | | |
| m (1/11) | 19, 56, 94 | 26, 80, 53 | 69, 34, 171 | |
| T (1/17) | 29c, 58, 29 | 41c, 82, 41 | -, 104, 53 | 67, -, 65 |

Table 3. Equal temperaments derived from a sample of 5-limit approximant ratio matches

The table includes a selection of well-known, low-cardinality equal temperaments which provide good approximations to 5-limit just intonation. 53edo, which is particularly accurate as judged on this criterion, appears twice in the table.

In the case of 29edo and 41edo the meantone and schismic versions of the temperament differ in the number of steps assigned to the major third. The 'meantone' versions (marked with the letter 'c') can alternatively be regarded as 3-limit temperaments, in which guise they also appear in Table 2. All other temperaments in the table are defined by patent vals.

The octave cardinalities are described as 'minimal' because of the possibility of integer subdivision. Note, however, that 29edo and 58edo (as defined by 5-limit patent vals) are not related in this simple way, since they differ in their tuning of major thirds.

The three temperaments represented by the number 12 are the same: in this case the categorisation as meantone, diaschismic or schismic does not yield distinct temperaments since 12edo is a member of all these families.

The two dashes in Table 3 indicate incompatible requirements – for example that the major third and the large tone should be tuned to a ratio of both 17:9 (approximant-matching) and 2:1 (meantone family).

The sets of numbers appearing in the cells of Table 3 carry suggestions of regular patterns, which we shall now investigate by considering the complete set of ETs for which given pair of source intervals are both approximant-matched and fairly accurately tuned.

Suppose the source intervals have approximants $v_1 = j_1/g_1$ and $v_2 = j_2/g_2$ expressed as reduced fractions. Approximant-matching equates their tempered gauges (here expressed in step units):

$$\hat{G}'_1 = \frac{\hat{J}'_1}{v_1} = \frac{\hat{J}'_2}{v_2} = \hat{G}'_2 \qquad (34)$$

which implies

$$\hat{J}'_1 j_2 g_1 = \hat{J}'_2 j_1 g_2 \qquad (35)$$

$$\hat{J}'_1 = \frac{m\, j_1\, g_2}{\mathrm{GCD}(j_1, j_2)\, \mathrm{GCD}(g_1, g_2)} \qquad (36)$$

where *m* is some positive integer. In anticipation of the analysis of *bimodular commas*, it is convenient to express this result in terms of a *rational multiplier* $b_\mathrm{m}$ for the interval pair $(J_1, J_2)$:

$$b_{\mathrm{m}}(J_1,J_2) = \frac{\mathrm{LCM}(j_1,j_2)}{\mathrm{GCD}(g_1,g_2)} = \frac{j_1\, j_2}{\mathrm{GCD}(j_1,j_2)\,\mathrm{GCD}(g_1,g_2)} \tag{37}$$

in terms of which

$$\hat{J}_1' = \frac{m\, b_m(J_1,J_2)}{v_2} \tag{38}$$

It then follows that

$$\hat{G}_1' = \frac{\hat{J}_1'}{v_1} = \frac{m\, b_{\mathrm{m}}(J_1,J_2)}{v_1 v_2} \tag{39}$$

If $J_1$ and $J_2$ are reasonably accurately tuned the temperament step size s is approximately

$$s = \frac{J_1}{\hat{J}_1'} \approx \frac{v_1}{\hat{J}_1'} = \frac{1}{\hat{G}_1'} \tag{40}$$

and the number of steps representing the octave is therefore approximately

$$\hat{\mathrm{o}} = \frac{\mathrm{o}}{s} \approx \frac{1}{2}\ln 2\ \hat{G}_1' = \frac{\frac{1}{2}\ln 2\ \ m\, b_{\mathrm{m}}(J_1,J_2)}{v_1 v_2} \tag{41}$$

Thus the octave cardinalities ô of the temperaments generated from $J_1$ and $J_2$ can be expected to cluster round a sequence of values which are integer multiples of a number close to $\frac{1}{2}\ln 2\,\frac{b_{\mathrm{m}}(J_1,J_2)}{v_1 v_2}$. This is borne out by the following examples.

The following examples explore some of these temperament groups in more detail.

***Example: Aristoxenean equal temperaments***

Temperaments which approximant-match f and F (and consequently temper out the Pythagorean comma) have

$$\hat{\mathrm{o}} \approx \frac{1}{2}\ln 2\ \ m\,/(\tfrac{1}{5}.\tfrac{1}{7}) = 12.13m \qquad (m = 1, 2, \ldots) \tag{42}$$

These temperaments have octave cardinalities which are multiples of 12.

***Example: Escapade equal temperaments***

Temperaments which approximant-match f and M (and consequently temper out the escapade comma) have

$$\hat{\mathrm{o}} \approx \frac{1}{2}\ln 2\ \ m\,/(\tfrac{1}{7}.\tfrac{1}{9}) = 21.83\, m \qquad (m = 1, 2, \ldots) \tag{43}$$

$\approx$ 21.83, 43.67, 65.50, ...

Temperaments exploiting this match include 22edo, 43edo, 44edo, 65edo and 66edo.

***Example: meantone equal temperaments***

Temperaments which approximant-match f and $m_7$ (and consequently temper out the syntonic comma) have

$$\hat{\mathrm{o}} \approx \frac{1}{2}\ln 2\ \ m\,(\tfrac{2}{7})/(\tfrac{1}{7}.\tfrac{2}{7}) = 2.43\, m \qquad (m = 1, 2, \ldots) \tag{44}$$

$\approx$ 2.43, 4.85, 7.27, 9.70, 12.13, 14.55, 16.98, 19.41, 21.83, 24.26, 26.69, 29.11, 31.54, ...

These are meantone temperaments, which are supported by the following edos and multiples thereof: 5, 7, 12, 17, 19, 22(meantone), 26, 29(meantone) and 31.

## Errors in bimodular approximants

An analysis of the errors in bimodular approximants provides a useful basis for understanding their relationships with equal temperaments and commas.

### Absolute and fractional errors in bimodular approximants

Errors in bimodular approximants can be analysed using the Taylor series for $\tanh^{-1} v$ and $\tanh J$, from which the following expressions are derived:

$$\frac{1}{G} = \frac{v}{J} = 1 - \tfrac{1}{3}v^2 - \tfrac{4}{45}v^4 - O(v^6) \tag{45}$$

$$= 1 - \tfrac{1}{3}J^2 + \tfrac{2}{15}J^4 - O(J^6) \tag{46}$$

$$G = \frac{J}{v} = 1 + \tfrac{1}{3}v^2 + \tfrac{1}{5}v^4 + O(v^6) \tag{47}$$

$$= 1 + \tfrac{1}{3}J^2 - \tfrac{1}{45}J^4 + O(J^6) \tag{48}$$

From eqn 45, the fractional error introduced by replacing the interval $J$ by its approximant $v$ is

$$\varepsilon_{\text{int}}(J) = \frac{v}{J} - 1 = -\tfrac{1}{3}v^2 - \tfrac{4}{45}v^4 - O(v^6) \tag{49}$$

In the following analysis we shall work to third order in $v$ and take the term $O(v^4)$ as read. Accordingly we write

$$\varepsilon_{\text{int}}(J) \approx -\tfrac{1}{3}v^2 \tag{50}$$

Table 4 shows the errors for some low-order 5-limit superparticular intervals expressed in cents and as fractions of the interval, with the fractional error estimates –⅓ $v^2$ shown for comparison in the last column.

| **Interval** | **Freq. ratio** | **Approxi-mant $v$ (dNp)** | **Approxi-mant $v$ (cents)** | **Just interval $J$ (cents)** | **Error (cents)** | **Fractional error $\varepsilon_{\text{int}}(J)$** | **Fractional error estimate** |
|---|---|---|---|---|---|---|---|
| Octave | 2/1 | 1/3 | 1154.156 | 1200.000 | -45.844 | -0.03820 | -0.03704 |
| Perfect fifth | 3/2 | 1/5 | 692.494 | 701.955 | -9.461 | -0.01348 | -0.01333 |
| Perfect fourth | 4/3 | 1/7 | 494.638 | 498.045 | -3.407 | -0.00684 | -0.00680 |
| Major third | 5/4 | 1/9 | 384.719 | 386.314 | -1.595 | -0.00413 | -0.00412 |
| Minor third | 6/5 | 1/11 | 314.770 | 315.641 | -0.871 | -0.00276 | -0.00275 |
| Large tone | 9/8 | 1/17 | 203.675 | 203.910 | -0.235 | -0.00115 | -0.00115 |
| Small tone | 10/9 | 1/19 | 182.235 | 182.404 | -0.169 | -0.00092 | -0.00092 |
| Semitone | 16/15 | 1/31 | 111.693 | 111.731 | -0.039 | -0.00035 | -0.00035 |
| Chroma | 25/24 | 1/49 | 70.663 | 70.672 | -0.010 | -0.00014 | -0.00014 |

Table 4. Approximant errors for low-order 5-limit superparticular intervals

The accumulated approximant error for all superparticular intervals is

$$\lim_{N\to\infty}\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2N-1} - \frac{1}{2}\ln N\right] \tag{51}$$

$$= \tfrac{1}{2}\gamma + \ln 2 - 1 = -0.018245\ldots \tag{52}$$

dineper, or –63.173 cents, where

$\gamma$ = 0.57721566... is the Euler-Mascheroni constant

About 73% of this error is accounted for by the error in the octave.

In view of the simplicity of the expression defining approximants, their accuracy is remarkable, though, as Table 4 shows, it is not sufficient for the precise calculation of absolute interval sizes.

The main value of approximants is in identifying approximate whole-number ratios between just intervals. For a pair of similarly sized intervals this ratio is more accurately approximated than the intervals themselves, due to a partial cancellation of errors. In the context of equal temperaments, intervals derived from approximants can also benefit from renormalisation – the adjustment of the scale to produce an exact match to an interval such as the octave.

## Error in the ratio of two approximants

Here we examine the error involved in estimating the ratio of two intervals using the ratio of their approximants. The fractional error in replacing the true interval ratio $J_2/J_1$ with the approximant ratio $v_2/v_1$ is found (using eqns 45 and 47) to be

$$\varepsilon_{\text{ratio}}(J_1, J_2) = \frac{v_2/v_1}{J_2/J_1} - 1 = \frac{v_2}{J_2}\frac{J_1}{v_1} - 1 \tag{53}$$

$$\approx \left(1 - \tfrac{1}{3}v_2^2\right)\left(1 + \tfrac{1}{3}v_1^2\right) - 1 \tag{54}$$

$$= -\tfrac{1}{3}\left(v_2^2 - v_1^2\right) = -\tfrac{1}{3}(v_2 + v_1)(v_2 - v_1) \tag{55}$$

to third order in $v_1$ and $v_2$.

$\varepsilon_{\text{ratio}}(J_1, J_2)$ can also be expressed in terms of $J_1$ and $J_2$ using eqn 48, and this yields a somewhat more accurate approximation because of the smaller size of the neglected fourth order terms:

$$\varepsilon_{\text{ratio}}(J_1, J_2) \approx -\tfrac{1}{3}\left(J_2^2 - J_1^2\right) = -\tfrac{1}{3}(J_2 + J_1)(J_2 - J_1) \tag{56}$$

A comparison of eqns 50 and 55 illustrates how, in estimates based on approximants, partial error cancellation produces a smaller fractional error for interval ratios than for absolute intervals. The error in the ratio estimate increases with both interval sum and interval difference.

## Error when a pair of approximants are scaled to tune their sum pure

Suppose we take the approximants $v_1$ and $v_2$ of a pair of intervals $J_1$ and $J_2$ and scale them both by the same factor (thus preserving their ratio) in such a way as to tune their sum pure. The error in the smaller interval is then found to be

$$\varepsilon_{\text{norm}}(J_1, J_2) \approx \tfrac{1}{3} v_1 v_2 (v_2 - v_1) \tag{57}$$

and the error in the larger interval is just the negative of this.

As with $\varepsilon_{ratio}(J_1, J_2)$, the analogous expression in terms of $J_1$ and $J_2$ is somewhat more accurate:

$$\varepsilon_{\text{norm}}(J_1, J_2) \approx \tfrac{1}{3} J_1 J_2 (J_2 - J_1) \tag{58}$$

Matches of this type, which we shall term *sum-normalised*, tend to produce high accuracy because the residual error after normalisation is shared equally between the two intervals.

### *Example: 12edo approximant match* (f,F)

12edo matches the interval pair (f, F) with $(v_1, v_2) = (1/7, 1/5)$, tuning the sum interval (the octave) pure. The error estimate (eqn 57) for the perfect fourth (which is the negative of the error estimate for the perfect fifth) is

$$\varepsilon_{\text{norm}}(\text{f},\text{F}) \approx \frac{1}{3}\left(\frac{1}{7}\right)\left(\frac{1}{5}\right)\left(\frac{1}{5}-\frac{1}{7}\right) = 0.0005442\ldots \text{ dNp} \approx 1.88 \text{ cents}$$

(or 1.98 cents using the more accurate eqn 58). The true figure is 1.96 cents.

### Error when a pair of approximants are scaled to tune the larger interval pure

Suppose now we scale $v_1$ and $v_2$ in such a way that the larger interval $J_2$ is tuned pure. Then the error in the smaller interval is, from eqn 55

$$\varepsilon_2(J_1,J_2) \approx \frac{1}{3} v_1(v_2 + v_1)(v_2 - v_1) \quad (59)$$

Alternatively

$$\varepsilon_2(J_1,J_2) \approx \frac{1}{3} J_1(J_2 + J_1)(J_2 - J_1) \quad (60)$$

Matches of this type have lower accuracy because the error is entirely carried by the smaller interval.

***Example: 12edo approximant matches* (M,o) *and* ($M_6$,o)**

12edo matches (M,o) with $(v_1, v_2)$ = (1/9, 1/3), and ($M_6$,o) with $(v_1, v_2)$ = (1/4, 1/3), in each case tuning the larger interval (o) pure. The error estimate (eqn 58) for the major third is

$$\varepsilon_2(\text{M},\text{o}) \approx \frac{1}{3}\left(\frac{1}{9}\right)\left(\frac{1}{3}+\frac{1}{9}\right)\left(\frac{1}{3}-\frac{1}{9}\right) \approx 0.003658\ldots \text{ dNp} \approx 12.67 \text{ cents}$$

(or 13.86 cents using the more accurate eqn 60). The true figure is 13.69 cents.

The error estimate for the major sixth (or minus the error estimate for the minor third) is

$$\varepsilon_2(\text{M}_6,\text{o}) \approx \frac{1}{3}\left(\frac{1}{4}\right)\left(\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{3}-\frac{1}{4}\right) \approx 0.004051\ldots \text{ dNp} \approx 14.03 \text{ cents}$$

(or 16.18 cents using the more accurate eqn 60).The true figure is 15.64 cents.

## Bimodular commas

As a consequence of the near-rational interval relationships implied by approximants, any pair of source intervals can be used to define a comma.

Given two intervals $J_1$ and $J_2$ (with $J_1 < J_2$) and their approximants $v_1$ and $v_2$, we define the *bimodular residue* as the difference between their gauges:

$$b_r(J_1,J_2) = G_2 - G_1 = \frac{J_2}{v_2} - \frac{J_1}{v_1} \quad (61)$$

Since for source intervals of ordinary size gauges are close to unity, $b_\text{r}(J_1,J_2)$ will typically be small. Using eqn 47 we find (to third order)

$$b_r(J_1,J_2) \approx \frac{1}{3}(v_2^2 - v_1^2) = \frac{1}{3}(v_2 + v_1)(v_2 - v_1) \quad (62)$$

This is identical, apart from the sign, to the third order expression obtained above for $\varepsilon_{\text{ratio}}(J_1,J_2)$, and this can be understood by noting that

$$\varepsilon_{\text{ratio}}(J_1,J_2) = \frac{v_2}{J_2}\frac{J_1}{v_1} - 1 = \frac{v_2}{J_2}\left(\frac{J_1}{v_1} - \frac{J_2}{v_2}\right) = -\frac{v_2}{J_2}\, b_\text{r}(J_1,J_2) \quad (63)$$

where the multiplier $v_2/J_2$ introduces only a fourth order correction.

The analogous approximation for $b_\text{r}(J_1,J_2)$ in terms of $J_1$ and $J_2$ is somewhat more accurate:

$$b_r(J_1,J_2) \approx \frac{1}{3}(J_2^2 - J_1^2) = \frac{1}{3}(J_2 + J_1)(J_2 - J_1) \quad (64)$$

We take as an example the bimodular residue for the combination of the fourth and the fifth, which is more familiar as the Pythagorean comma:

$$b_{\mathrm{r}}(\mathrm{f},\mathrm{F}) \approx 5\mathrm{F} - 7\mathrm{f} = \tfrac{1}{3}(\mathrm{f}+\mathrm{F})(\mathrm{F}-\mathrm{f}) \tag{65}$$

This estimate evaluates to 23.56 cents, close to the accurate value of 23.46 cents.

By an accident of scaling (100/(3×3462.468) = 0.009627 ≈ 1/100), if $J_1$ and $J_2$ are expressed in 12edo semitones, $(J_1 + J_2)(J_1 - J_2)$ is close to the size of the bimodular residue in cents. For example, setting $J_1$ = f ≈ 5, $J_2$= F ≈ 7 semitones gives $b_{\mathrm{r}}(J_1,J_2)$ ≈ (5+7)×(7–5) = 24 cents. This provides a quick method for estimating bimodular residue sizes.

Commas are conventionally defined as intervals with rational frequency ratios for which the prime exponents do not all share a common factor. Accordingly we convert the bimodular residue to a *bimodular comma*, $b(J_1,J_2)$, in which this property is achieved by applying a suitably chosen rational multiplier $b_{\mathrm{m}}(J_1,J_2)$:

$$b(J_1,J_2) = \mathrm{b_m}(J_1,J_2)\,\mathrm{b}_r(J_1,J_2) \tag{66}$$

If the approximants are expressed as irreducible fractions

$$v_1 = \frac{j_1}{g_1}, \qquad v_2 = \frac{j_2}{g_2}$$

then

$$b_{\mathrm{m}}(J_1,J_2) = \frac{\mathrm{LCM}(j_1\,,j_2)}{\mathrm{GCD}(g_1,g_2\,)} \tag{67}$$

(In rare instances a further multiplier is required to avoid a common factor in the prime exponents; b(T,o) is such an instance, where a factor of ½ must be applied.)

As an example we consider the pair of intervals:

| | | |
|---|---|---|
| $J_1$ = <u>4/3</u> = f, | $v_1$ = 1/7 | (perfect fourth) |
| $J_2$ = <u>9/5</u> = $\mathrm{m}_7$, | $v_2$ = 2/7 | (minor seventh) |

for which

$$b_{\mathrm{r}}(J_1,J_2) = \frac{J_2}{v_2} - \frac{J_1}{v_1} = \tfrac{7}{2}\,\mathrm{m}_7 - 7\mathrm{f} \tag{68}$$

$$b_{\mathrm{m}}(J_1,J_2) = \frac{\mathrm{LCM}(1,2)}{\mathrm{GCD}(7,7)} = \frac{2}{7} \tag{69}$$

$$b(J_1,J_2) = \mathrm{b_m}(J_1,J_2)\,\mathrm{b}_r(J_1,J_2) = \mathrm{m}_7 - 2\mathrm{f} \tag{70}$$

The statement that a bimodular comma $b(J_1,J_2)$ (or equivalently the corresponding bimodular residue) is tempered out by an equal temperament is equivalent to the statement that $J_1$ and $J_2$ are approximant-matched in the temperament. This can be seen by noting (using eqn 19) that

$$b'(J_1,J_2) = 0 \quad \Leftrightarrow \quad G_1' = G_2' \tag{71}$$

Many commas of the bimodular type feature in established temperament theory, which is to say that there are many instances of approximant-matched intervals in known temperaments. In the example above, $b(f,\mathrm{m}_7)$ is the syntonic comma, c. Some more examples are listed below.[5]

***Examples – 3-limit approximant commas:***

*b*(f, o) = |-11 7> = <u>2187/2048</u> (Pythagorean chroma, 113.685 cents)

*b*(F, o) = |8 -5> = <u>256/243</u> (limma, 90.225 cents)

*b*(T, o) = *b*(F, $m_7^-$) = |27 -17> (17-tone comma, 66.765 cents)
*b*(T, F) = |46 -29> (sub-limma, 43.304 cents)
*b*(f, F) = |-19 12> = 531441/524288 (comma of Pythagoras, 23.460 cents)
*b*(T, f) = |65 -41> (41-tone comma, 19.845 cents)

***Examples – 5-limit approximant commas:***
*b*($M_6$, o) = |3 4 -4> = 648/625 (major diesis, 62.565 cents)
*b*(M, $M_6$) = |18 -4 -5> = 262144/253125 (passion comma, 60.611 cents)
*b*(f, $M_6$) = |-14 3 4> = 16875/16384 (negrisma, 51.12 cents)
*b*(M, o) = |7 0 -3> = 128/125 (diesis, 41.059 cents)
*b*(m, F) = |-16 -6 11> (sycamore comma, 37.721 cents)
*b*(M, F) = |13 5 -9> (valentine comma, 32.952 cents)
*b*(F, $M_6$) = |5 -9 4> = 20000/19683 (minimal diesis, 27.660 cents)
*b*(f, $m_7$) = |-4 4 -1> = 81/80 (syntonic comma, 21.506 cents)
*b*(M, f) = |32 -7 -9> (escapade comma, 9.492 cents)
*b*(t, $D_4$) = |8 14 -13> (parakleisma, 5.292 cents)
*b*(m, M) = |-29 -11 20> (gammic comma, 4.769 cents)
*b*(X, f) = |23 6 -14> (semisuper comma, 3.338 cents)
*b*(s, t) = |-104 -7 50> (2.001 cents)
*b*(c, X) = |71 -99 37> (raider comma, 0.062 cents)
*b*(c, D) = |161 -84 -12> (atom of Kirnberger, 0.015 cents)

***Examples – 7-limit approximant commas:***
*b*(X, 7/3) = |2 -3 0 1> = 28/27 (Archytas' 1/3 tone, 62.961 cents)
*b*(7/5, o) = |1 0 2 -2> = 50/49 (jubilisma, 34.976 cents)
*b*(m, 7/4) = |-5 -3 3 1> = 875/864 (keema, 21.902 cents)
*b*(7/5, $M_6$) = |0 -2 5 -3> = 3125/3087 (gariboh comma, 21.181 cents)
*b*(t, 12/7) = |-3 11 -5 -1> = 177147/175000 (21.111 cents)
*b*(9/7, $M_6$) = |0 -5 1 2> = 245/243 (octarod, 14.191 cents)
*b*(7/6, $m_6$) = |6 3 -1 -3> = 1728/1715 (Orwell comma, 13.074 cents)
*b*(8/7, 7/5) = |-15 0 -2 7> 823543/819200 (quince, 9.154 cents)
*b*(8/7, F) = |-10 1 0 3> = 1029/1024 (gamelisma, 8.433 cents)
*b*(9/7, 7/5) = |0 -8 -3 7> 823543/820125 (7.200 cents)
*b*(M, 7/5) = |6 0 -5 2> 3136/3125 (parahew, 6.083 cents)
*b*(T, 10/7) = |10 -6 1 -1> 5120/5103 (hemifamity, 5.758 cents)
*b*(s, 35/27) = |-16 1 5 1> 65625/65536 (tertiapoint, 2.349 cents)
*b*($s^+$, 7/6) = |-1 -7 4 1> 4375/4374 (ragisma, 0.396 cents)

The source interval pairs with frequency ratios

$$(r_1, r_2) = \left(\frac{m+1}{m-1}, \frac{m+2}{m-2}\right) \qquad (m = 3, 4, 5, \dots) \qquad (72)$$

produce an infinite sequence of bimodular commas with frequency ratios

$$\left(\frac{m+2}{m-2}\right)/\left(\frac{m+1}{m-1}\right)^2 = \frac{m(m^2-3)+2}{m(m^2-3)-2} \qquad (m = 3, 4, 5, \dots) \qquad (73)$$

The sequence consists mostly of superparticular commas (with exceptions when $m$ is a multiple of 4). Its first few members are:

| | | | |
|---|---|---|---|
| 5/4 (major third) | 27/25 (semitone maximus) | 28/27 (Archytas's 1/3 tone) | |
| 50/49 (jubilisma) | 81/80 (syntonic comma) | 245/243 (octarod, minor BP diesis) | |
| 176/175 (valinorsma) | 243/242 (neutral third comma) | 325/324 (marveltwin) | 847/845 (cuthbert) |
| 540/539 (swetisma) | 676/675 (island comma) | 833/832 | 2025/2023 |
| 1216/1215 (Eratosthenes) | 1445/1444 | 1701/1700 | 3971/3969 |
| 2300/2299 | 2646/2645 | 3025/3024 (lehmerisma) | 6877/6875 |
| 3888/3887 | 4375/4374 (ragisma) | | |

Other examples of superparticular bimodular commas are the perfect fourth and the large tone:

$b$(F, 9) = 4/3 = f, $\qquad$ $b$(o, $P_{12}$) = 9/8 = T

A bimodular comma has a prime-limit which is the larger of the prime-limits of its source intervals, and in tonal space it is coplanar with the source intervals and the origin.

The sizes of bimodular commas can be estimated from the rule of thumb for bimodular residues derived above (in the simplest and therefore most important cases the bimodular residue and comma are often one and the same). For pairs of source intervals of octave size and less this yields a range of zero to 144 cents, with an average of 48 cents. This result provides a degree of explanation for the size range of the commas which have been found to be useful in temperament theory.

When $b_m$ has a denominator greater than one the bimodular comma is both smaller and of lower complexity than the corresponding bimodular residue – both desirable qualities in the context of temperaments. This effect has already been demonstrated in the case of the syntonic comma. Another striking example is provided by the bimodular comma formed from the syntonic comma (c = 81/80) and the dieses (D = 128/125).Using the approximants given in Table 1 we find

$b_r$(c,D) = (11×23/3)D – (7×23)c

$b_m$(c,D) = LCM(3,1) / GCD(11×23,7×23) = 3/23

$b$(c,D) = $b_m$(c,D) $b_r$(c,D) = 11D – 21c = $2^{161} / 3^{84}\, 5^{12}$

The factor 23 shared by the denominators of these source intervals' approximants, combined with the small size of those intervals, means that $b$(c,D) is both extremely small (a mere 0.01536 cents) and expressible simply in terms of other small 5-limit intervals. In terms of the syntonic comma (c), the Pythagorean comma (p = 3c – D) and the schisma (σ = p – c),

$b$(c,D) = 12c – 11p = c – 11σ = p – 12σ

The small size of $b$(c,D) means that these intervals, together with the diesis and the diaschisma (d = c – σ), are tuned very accurately by the steps of a miniature equal temperament (of which 612edo is one realisation) approximating a scale of schismas or grads (1 grad = p/12):

σ : d : c : p : D ≈ 1 : 10 : 11 : 12 : 21

The interval we have notated as $b$(c,D) has been studied since the 18th century and is known as the *atom of Kirnberger*. An explanation for its small size, based on arguments similar to those presented above, has been published by Page.[6]

A factor 23 is in fact present in either the numerator or the denominator of the approximant of *every* 5-limit interval which vanishes in 12edo if the temperament is defined using the patent val. A proof of this can be constructed from i) the fact that every such interval can be expressed as a linear combination of c and D with integer coefficients, and ii) the rule for combining approximants for interval sums and differences (eqn 23). It is also straightforward to show that this property is shared by any interval which is tempered to a multiple of 11 steps by 12edo. For example $J = \underline{24}$, which has $v = 23/25$ and is tempered to 55 steps.

This result relates to the following theorem, which is stated here without proof:

> When plotted on a tonal lattice based on any set of primes, points for which $jg = 0$ (mod $p$), where $j/g$ is the reduced approximant and $p$ is an odd prime not in the basis set, lie on a regular sub-lattice having a unit cell with hypervolume $(p-1)/2$ or (rarely) a factor thereof.

In the above example $p = 23$, $(p-1)/2 = 11$, and planes of the sub-lattice happen to align with the pitch contours of patent 12edo.

## Graphing matched approximant ratios in ETs: Jacana diagrams

A space of real numbers $(g, j)$, with $g > |j|$, in which $v = j/g$ is interpreted as a bimodular approximant, provides a natural coordinate system for studying equal temperaments (with axes are rotated by 45 degrees relative to the $(d, n)$ system illustrated in Figure 1). The subspaces in which $j$ and $g$ are confined to integers or rationals include representations of all just intervals, and the gradient $(v = j/g)$ of a line joining the origin to the point $(g, j)$ is the approximant of the corresponding interval.

Vertical lines of points in this space, $(g, j)$ ($j$ = 0, 1, 2,...), correspond to the sequences of approximately equally spaced intervals that we have termed *near-equal progressions*.

Near-equal progressions are related to a general principle which can be stated thus:

> Just intervals which are tempered to $j$ steps of an equal temperament correspond to points with rational $g$ coordinate and integer $j$ coordinate in a certain region of $(g, j)$ space.

Suppose that an equal temperament is defined by dividing an interval $J_o$, such as the octave, into $j_o$ equal steps. $j_o$ is then the *cardinality* of the ET, and its step size is

$$s = J_o/j_o \tag{74}$$

The interval $J_o$, with approximant $v_o$, is represented in $(g, j)$ space as a point $(g_o, j_o)$, where

$$g_o = j_o/v_o \tag{75}$$

The general interval $J$, with approximant $v$, is tempered to $j$ steps of the temperament:

$$\hat{J}' = j \tag{76}$$

$$J' = js \tag{77}$$

and the tempered interval $J'$ is represented by a point $(g', j')$, where

$$g = j/v \tag{78}$$

is the interval's tempered gauge $(\hat{G}')$.

The temperament error, measured in step units, is

$$\hat{\varepsilon} = \frac{J'}{s} - \frac{J}{s} = \hat{J}' - \hat{J} = j - \hat{J} \tag{79}$$

where $\hat{J}$ is the true interval size in step units:

$$\hat{J} = \frac{\tanh^{-1} v}{s} \tag{80}$$

Thus

$$\hat{\varepsilon} = j - \frac{\tanh^{-1} v}{s} \tag{81}$$

$$j = \frac{\tanh^{-1} v}{s} + \hat{\varepsilon} \tag{82}$$

We assume in this analysis that $J'$ is tuned to the nearest whole step of the temperament (while acknowledging that other ET tuning policies, including those based on vals, are possible). Under this assumption the magnitude of the temperament error is never more than half a step:

$$-\frac{1}{2} < \hat{\varepsilon} \leq \frac{1}{2} \tag{83}$$

where equality has been excluded at the lower end of the range in order to define the temperament unambiguously.

Points representing tempered intervals with an error of no more than half a step will be contained in a region – the *tempered domain* – bounded on the upper left and lower right by the two branches of the $\hat{\varepsilon} = -\frac{1}{2}$ *curve*

$$j = \frac{\tanh^{-1} v}{s} - \frac{1}{2} \tag{84}$$

and on the upper right and lower left by the two branches of the $\hat{\varepsilon} = \frac{1}{2}$ *curve*

$$j = \frac{\tanh^{-1} v}{s} + \frac{1}{2} \tag{85}$$

where $v$ runs between –1 and 1 and the $g$ coordinate is given by

$$g = j/v \tag{86}$$

Here $j$ is treated as a continuous variable, but the error limit will apply to all points in the tempered domain, including the subset of points having integer $j$ which constitute our main focus.

Lines of zero error ($\hat{\varepsilon} = 0$) are defined by

$$j = \frac{\tanh^{-1} v}{s} \tag{87}$$

This condition is met trivially on the line $j = 0$ and on the *zero error curve* defined by

$$g = \frac{j}{v} = \frac{j}{\tanh(js)} \tag{88}$$

The two lines intersect at right angles at a saddle point of the error function located at $(g, j) = (g_s, 0)$, where

$$g_s = \frac{1}{s} \tag{89}$$

The borders of the tempered domain (shown in black) and the two zero error lines (shown in grey) are plotted for 12edo in Figures 3 and 4. The *Jacana diagram* (named for its passing resemblance to the foot of that bird) is a graphical representation of an equal temperament in which each interval is tuned to the nearest whole step. An interval $J$ with approximant $v$ is represented on the diagram by a line of slope $v$ passing through the origin. The number of steps ($\hat{J}' = j$) to which $J$ is tempered may be read off from the point at which this line intersects a horizontal line representing an integer $j$-value within the tempered domain.

The Jacana diagram is an aid to understanding interval relationships and errors in nearest-step equal temperaments within an approximant-based framework. It shows that approximant-matched intervals in an ET do not occur haphazardly, but form vertical lines lying within a contiguous domain of the $(g, j)$ plane, the central region of which contains a series of unbroken chains of matched intervals with consecutive integer $j$ values.

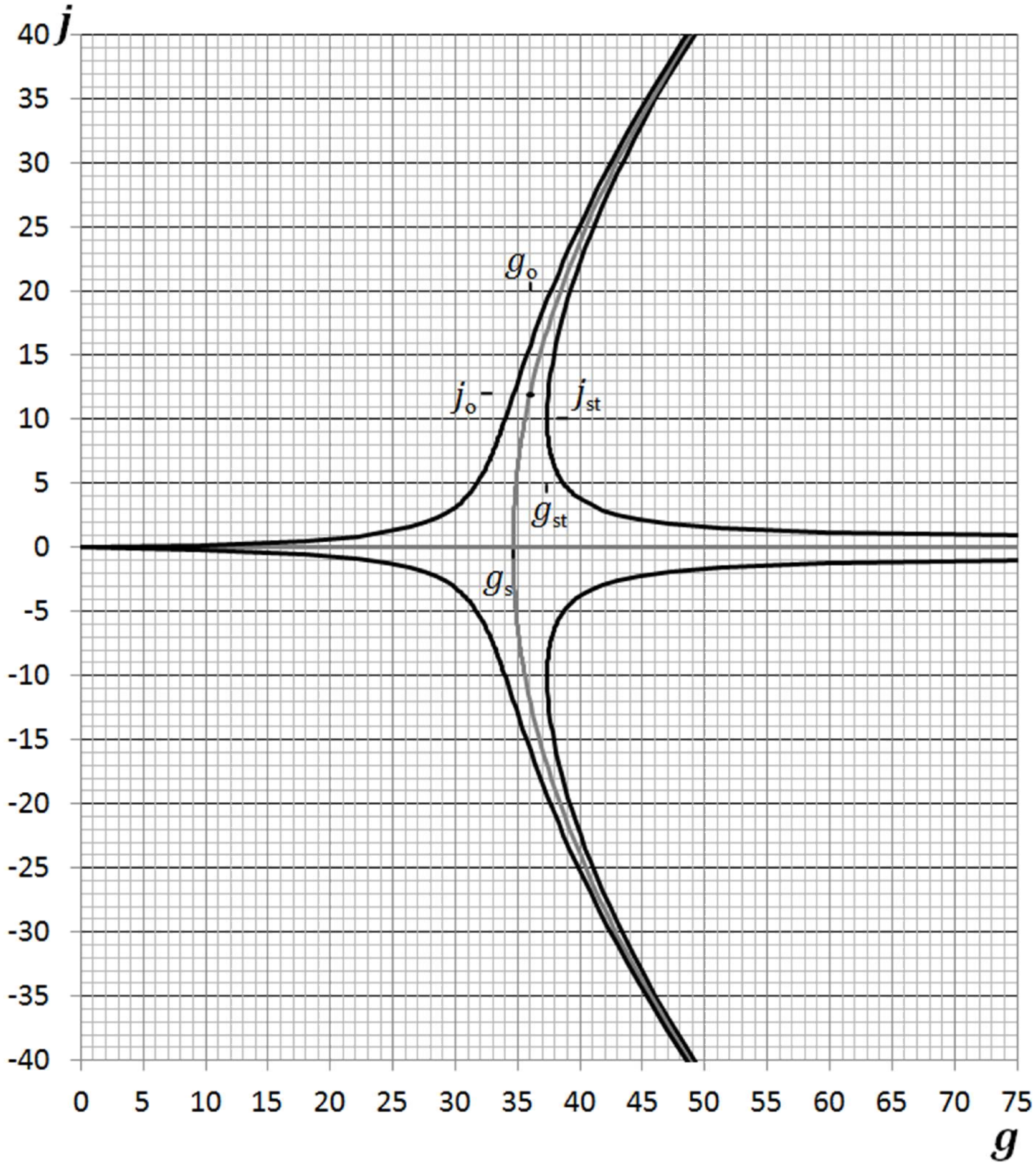


Figure 3. Jacana diagram for nearest-step 12edo

The following general properties of Jacana diagrams can be noted by inspection or proved without difficulty.

The tempered domain has four tapering 'toes' extending from a region centred on the point $(g_s, 0)$.

The gradient of the $\hat{\varepsilon} = -\frac{1}{2}$ curve at $j = 0$ is $\tanh(s/2)$, this being the approximant of half the step interval.

The line representing interval $J$ crosses the boundaries of the tempered domain at two points, which for $|v| \geq \tanh(s/2)$ are separated by distance 1 in the $j$ dimension and $1/|v|$ in the $g$ dimension. (Except where $|v|$ takes values near 0 or 1, therefore, $1/|v|$ gives a rough measure of the varying width of the toes which extend either side of the $g$ axis.)

Points lying on any vertical line $(g, j)$ $(j = 0, 1, 2, \ldots)$ lying within the tempered domain map to intervals which are approximant-matched in the temperament, with tempered gauge $g$ steps (where in cases of practical interest $g$ is often an integer). Instances of such matches in 12edo are examined below.

The tempered domain may be subdivided by plotting contours for errors of less than half a step (as in Figure 4). Contours for errors of ±1.5, ±2.5... steps may also be plotted, which divide the space into domains in each of which j (an integer determined as in the tempered domain) is offset from $\hat{J}'$ by ±1, ±2... In each of these domains every just interval is represented exactly once. For just intervals with reduced approximant $j_r/g_r$, the value of $g$ corresponding to integer $j$ will be an integer in every $j_r$'th domain.

The toes of the tempered domain are threaded by two zero-error lines, one horizontal (expressing the exact tuning of the unison), the other curving towards 45 degree asymptotes at large $|j|$, and approximated by

$$g \approx g_s + \tfrac{1}{3}j^2 s \qquad (v \ll 1) \tag{90}$$

$$g \approx |j|\left(1 + 2e^{-2js}\right) \qquad (v \approx 1) \tag{91}$$

The two zero-error lines cross at the point ($g_s$,0).

The number of steps, $\hat{J}'$, representing the true size of the interval $J$ may be read off from the $j$ coordinate of the point at which the line with slope $v$ intersects the zero error curve.

The point $(g_o, j_o)$ representing the interval which is normalised (tuned true) in the temperament lies on the $\hat{\varepsilon} = 0$ curve.

The form of the diagram depends only on the step size, $s$.

The zero error curve has the same shape for all ETs, scaling with $1/s$ in both dimensions.

$g_\mathrm{o}$ is somewhat greater than $g_s$ in fractional terms, as can be seen by noting that

$$\frac{g_\mathrm{o}}{g_s} = g_\mathrm{o}s = g_\mathrm{o}\frac{J_\mathrm{o}}{j_o} = \frac{J_\mathrm{o}}{v_o} = G_\mathrm{o} = 1 + \tfrac{1}{3}v_\mathrm{o}^2 + O(v_\mathrm{o}^4) \tag{92}$$

and therefore

$$g_\mathrm{o} - g_s = \tfrac{1}{3}g_s v_\mathrm{o}^2 + g_s O(v_\mathrm{o}^4) = \tfrac{1}{3}\left(\frac{j_\mathrm{o}}{J_o}\right)v_\mathrm{o}^2 + \left(\frac{j_\mathrm{o}}{J_o}\right)O(v_\mathrm{o}^4) \approx \tfrac{1}{3}j_\mathrm{o}v_\mathrm{o} \tag{93}$$

In the case of 12edo,

$$g_\mathrm{o} - g_s \approx \frac{1}{3} \times 12 \times \frac{1}{3} = \frac{4}{3}$$

The point $(g_{st}, j_{st})$ at which the $\hat{\varepsilon} = -\frac{1}{2}$ curve is stationary with respect to changes in $j$ is approximated by

$$g_\mathrm{st} = g_s + \left(\frac{3}{4}\right)^{2/3} g_s{}^{1/3} + \frac{1}{5}\left(\frac{3}{4}\right)^{4/3} g_s{}^{-1/3} + O(g_s{}^{-1}) \tag{94}$$

$$j_\mathrm{st} = \left(\frac{3}{4}\right)^{1/3} g_s{}^{2/3} + \frac{9}{20} + O(g_s{}^{-2/3}) \tag{95}$$

with $j_{st}$ and $g_{st}$ satisfying the exact relationship

$$j_\mathrm{st}{}^2 = g_\mathrm{st}(g_\mathrm{st} - g_\mathrm{s}) \tag{96}$$

In the case of 12edo,

$$(g_\mathrm{st}, j_\mathrm{st}) \approx (37.357, 10.102) \tag{97}$$

For ETs with cardinality $j_o$ greater than a critical value, $g_\mathrm{o}$ exceeds $g_\mathrm{st}$. The critical value is

$$j_\mathrm{o\_crit} = \frac{9\sqrt{3}}{4}\left(\frac{1}{J_\mathrm{o}^2} + \frac{1}{5}\right) + O(J_\mathrm{o}^2) \tag{98}$$

For $j_\mathrm{o}$ less than $j_{o\_\mathrm{crit}}$ the points $(g_o, j)$ lie within the tempered domain for all $|j| \le j_o$, and the corresponding intervals therefore form an approximant-matched set with tempered gauge $g_o$. For octave-based ETs ($J_\mathrm{o}$= o = 0.346574 ...) (99)

$$j_\mathrm{o\_crit} \approx 33.2 \tag{100}$$

Thus for octave-based ETs with cardinality 33 or less there is an approximant-matched set containing all steps of the temperament up to and including the octave (and in practice a little beyond).

For $j_o > j_{o\_\mathrm{crit}}$ this approximant-matched set splits into three disjoint portions (if negative intervals are included), but the central portion can be shown to include all $j$ satisfying

$$|j| \leq j_{\mathrm{lim}} = \frac{J_\mathrm{o}}{2(J_\mathrm{o} - v_\mathrm{o})} \tag{101}$$

where for octave-based ETs ($J_\mathrm{o}$= o = 0.346574..., $v_\mathrm{o}$ = 1/3),

$$j_{\mathrm{lim}} \approx 13.09 \tag{102}$$

Thus for octave-based ETs up to and including 33edo the approximant-matched set defined by the tempered gauge $g_o$ includes all steps up to the octave, and for higher cardinalities it includes at least the first 13 steps. In practice it is found that 34edo matches the first 17 steps, and the number of matches falls with increasing octave cardinality before levelling off at 13 steps for 54edo and above.

As a corollary of this result, 3steps of any octave-based temperament represent the interval whose approximant is the reciprocal of the octave cardinality (provided the cardinality is 3 or greater). Examples:

3 steps of 4edo make a major sixth, $M_6$ = 5/3 ($v$ = 1/4).

3 steps of 15edo make an 8/7 ($v$ = 1/15).

3 steps of 17edo make a large tone, T = 9/8 ($v$ = 1/17).

3 steps of 19edo make a small tone, t = 10/9 ($v$ = 1/19).

3 steps of 31edo make a diatonic semitone, s = 16/15 ($v$ = 1/31).

3 steps of 41edo make a 21/20 ($v$ = 1/41).

3 steps of 49edo make a chroma, X = 25/24 ($v$ = 1/49).

Points $(g, j)$ (where $g$ and $j$ are integers) lying close to the zero error curve represent intervals tuned with exceptional accuracy by the temperament, with an error of the order of one percent of the ET step. In the region where $g \approx g_s$ and $v << 1$ it follows from eqns 81 and 90 that

$$\frac{\mathrm{d}\hat{\varepsilon}}{\mathrm{d}j} \approx -\frac{2}{3}v^2 \tag{103}$$

For a given integer $g$, the point with integer $j$ nearest the zero error curve will have $|\Delta j| \leq \frac{1}{2}$, and the magnitude of the associated error (in step units) is therefore limited by

$$|\hat{\varepsilon}| \approx \frac{2}{3}v^2|\Delta j| <\approx \frac{1}{3}v^2 \tag{104}$$

For intervals smaller than an octave ($v$ < 1/3) the maximum error is about 0.037 steps and the mean error (based on uniformly distributed and independent $v$ and $|\Delta j|$) about 0.006 steps.

For the temperament $j_\mathrm{o}$-edo, sub-octave just intervals in this accurately tuned set have integer $g$ values between $g_s$ and $g_\mathrm{o}$. The number of such points is

$$\lfloor g_\mathrm{o} - g_\mathrm{s} \rfloor = \left\lfloor \left(\frac{1}{v_o} - \frac{1}{J_\mathrm{o}}\right) j_\mathrm{o} \right\rfloor = \left\lfloor \left(3 - \frac{2}{ln2}\right) j_\mathrm{o} \right\rfloor = \lfloor 0.11461 j_\mathrm{o} \rfloor \tag{105}$$

where $\lfloor x \rfloor$ denotes the *floor function*, which maps $x$ to the largest integer not greater than $x$.

The Benedetti height, $nd$, of these intervals is no greater than $g_o{}^2 = 9j_o{}^2$. Each has an octave-complement which is tuned equally accurately.

Examples include 41/29 in 12edo (5.995 steps), 5/3 ($M_6$) in 19edo (14.002 steps), 8/5 ($m_6$) in 31edo, 16/11 in 37edo (20.001 steps) and 3/2 ($P_5$) in 53edo (31.003 steps).

## Further reflections on 12edo

Figure 4 is a close-up of one toe of the Jacana diagram for 12edo. Low-complexity intervals are identified in font sizes reflecting their prime limit (5, 7 or 11), and the zero error lines and the boundaries of the tempered domain (error = ±50 cents) are supplemented with lines representing errors of ±25 cents. Intervals lying to the left and right of the zero error curve are tuned flat and sharp, respectively. The matched sets previously identified appear in vertical columns with $\hat{G}' = g$.

In the discussion so far, insights into the accuracy of 12edo as a 5-limit tuning have been obtained by considering its most important groups of 5-limit approximant ratio matches – those with $\hat{G}' = 35$ and 36. The second of these groups also includes an interval, 7/5, which casts light on the 7-limit characteristics of 12edo. We now examine these matches more closely.

As a first step we examine octave-based equal temperaments of low cardinality. Such temperaments can be derived by approximant-matching the octave to intervals whose approximants are reciprocals of small integers – intervals which have a reduced frequency ratio for which the difference between the numerator and denominator (the *degree of epimoricity*) is 1 or 2. As a consequence of this property the first eight of them – those with approximants 1/2, 1/3, 1/4,... 1/9 (i.e. those with denominators stopping 2 short of 11) – are all 7-limit intervals. Table 5 shows the ETs derived by approximant matching these intervals to the octave, with the error in the matched non-octave interval indicated in the last column.

The octave cardinalities increase in the sequence 2, 3, 4,... 9 except where reduced by common factors between the denominators of the matched approximants. The first two entries in the table are included for completeness and will serve no further purpose.

Returning to 12edo, we recall that its accuracy in the 3-limit is ensured by a sum-normalised match based on the source interval pair (f, F) with tempered gauge 35, while its 5 and 7-limit credentials are closely bound up with the matched set (M, 7/5, $M_6$, o) which has tempered gauge 36.

These sets can be related by noting that the first of them links a pair of consecutive superparticular intervals, 3/2 and 4/3. Such interval pairs have frequency ratios of the form

$$(r_1, r_2) = \left(\frac{m}{m-1}, \frac{m+1}{m}\right) \tag{106}$$

with approximants

$$(v_1, v_2) = \left(\frac{1}{2m-1}, \frac{1}{2m+1}\right) \tag{107}$$

Their sum has frequency ratio $(m+1)/(m-1)$ and approximant $1/m$.

Approximant-matching this pair of intervals and normalising their sum yields a temperament which divides the sum interval into $4m$ equal parts. In this temperament family the sum interval has tempered gauge $4m^2$ and the source intervals have tempered gauge $4m^2 - 1$.

In the case of 12edo, $m = 3$, the cardinality is $4m = 12$ and the tempered gauges are $4m^2$= 36 for the octave and $4m^2 - 1 = 35$ for the fourth and the fifth.

We note next that the octave cardinality (12) has factors 2, 3 and 4, making12edo a superset of 2edo, 3edo and 4edo. 12edo therefore inherits the approximant matches for these lower-cardinality temperaments, which, as we have seen, guarantee acceptable tuning for the major and minor thirds and sixths and the 7/5.

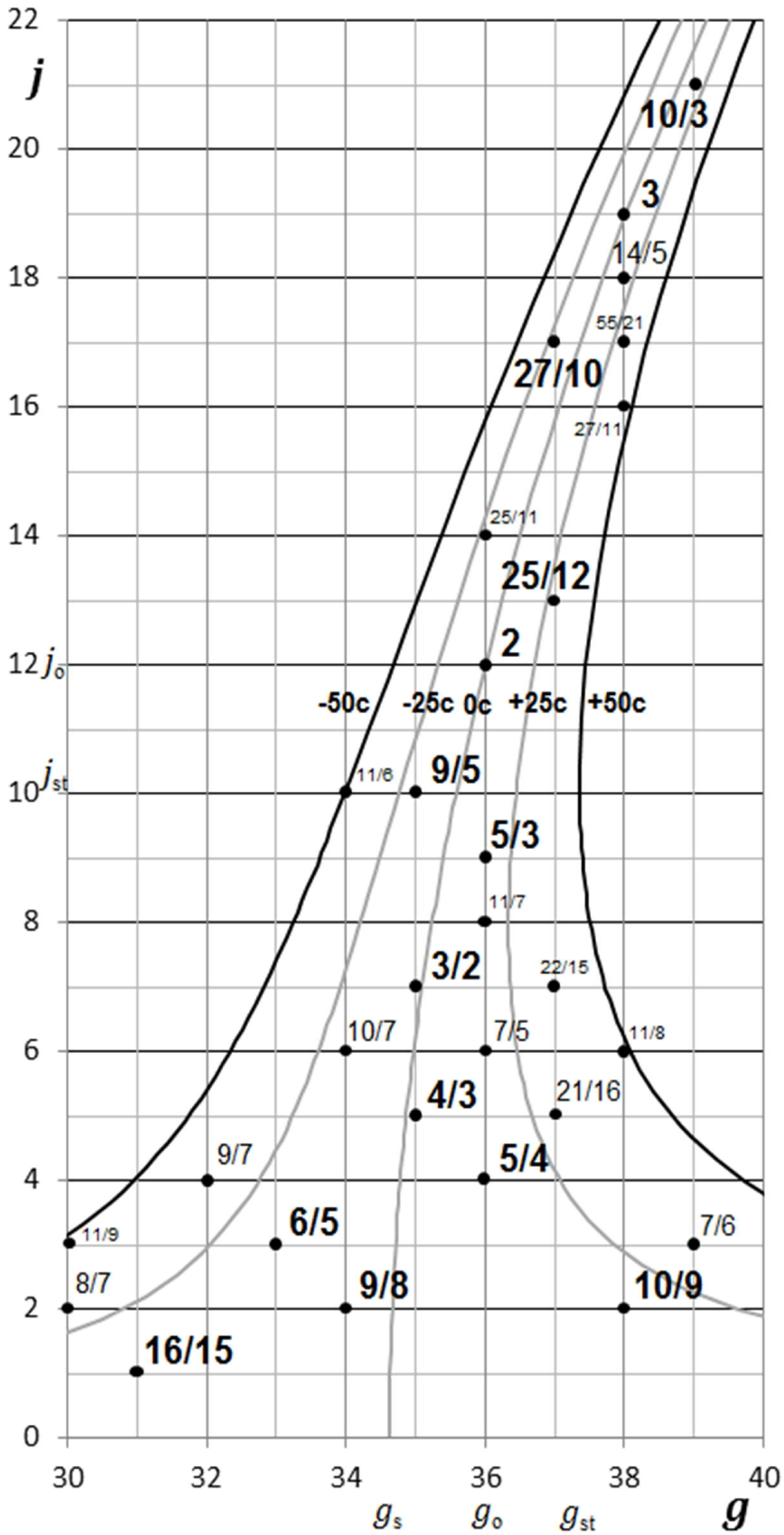


Figure 4. Jacana diagram for nearest-step 12edo (detail)

| *Interval* | *Approximant* | *Derived ET* | *Error (cents)* |
|---|---|---|---|
| Tritave = $P_{12}$ = 3/1 | 1/2 | 2edo | -102.0 |
| Octave = o = 2/1 | 1/3 | 1edo | 0.00 |
| Major sixth = $M_6$ = 5/3 | 1/4 | 4edo | 15.6 |
| Perfect fifth = F = 3/2 | 1/5 | 5edo | 18.0 |
| 7/5 | 1/6 | 2edo | 17.5 |
| Perfect fourth = f = 4/3 | 1/7 | 7edo | 16.2 |
| 9/7 | 1/8 | 8edo | 14.8 |
| Major third = M = 5/4 | 1/9 | 3edo | 13.7 |

Table 5. Low-cardinality octave-based ETs derived by approximant-matching with the octave

These arguments provide an explanation for the degree to which 12edo approximates the 3-limit (very accurately), the 5-limit (tolerably well) and the 7-limit (somewhat less well).

Finally, we identify two further members of the temperaments family parameterised by $m$. Setting $m = 4$ divides the major sixth into $4m = 16$ equal parts, with accurate fourths and major thirds. This is an adjusted version of 22edo. Setting $m = 5$ divides the fifth into $4m = 20$ equal parts, with accurate major and minor thirds. This is Carlos *gamma*, or adjusted 34edo.

## Christmas tree diagrams

The Christmas tree diagram provides another way of graphing the space of points $(g, j)$ to visualise the properties of an equal temperament.

This diagram plots $v = j/g$ against $g$ for a suitable range of positive integer values of $j$ and $g$, using logarithmic scales for both axes.

Points with the same approximant $v$, and therefore the same interval and frequency ratio, lie on a horizontal line, with interval size increasing up the page. Points on the horizontal line correspond to successive integer values of $j$, their $g$ values being multiples of the reduced approximant denominator. In the examples the points are labelled with the associated frequency ratios up to prime limit 13, and plotting is selective for intervals greater than the octave.

Points with the same $g$ value lie on vertical lines which correspond to vertical lines on the Jacana diagram (near-equal progressions).

Points with the same $j$ value lie on lines which slope from top-left to bottom-right.

The approximants $v$ of the set of points with $g \leq N$ form the Farey sequence of order $N$.

The diagram is decorated with coloured circles and lines to represent relationships in an equal temperament. In the examples the temperament is defined on the same convention as the Jacana diagram – the nearest-step basis.

Figure 5 is a Christmas tree diagram for 12edo. This example includes, on the right hand side, grey lines marking the boundaries of the tempered domain. Points within this domain are ringed with circles of four pastel colours representing tempered gauges of 34, 35, 36 and 38 steps, respectively. A vertical line in the same colour is drawn through the points associated with each tempered gauge.

To the left of these points, other points representing the same set of intervals are also ringed in the same or related colours.

A striking feature of the diagram is the symmetrical disposition of groups of coloured circles about the lines which form the 'trunk' of the tree. Circles of a particular colour lie at equal distances on either side of a 45 degree diagonal line of the same colour, the colour coding for the tempered gauge which the circled points share. Saturated colour is used for points which participate in this mirror symmetry and the associated mirror lines. The mirror line with tempered gauge $\hat{G}$ has the formula $g = \hat{G}'v$, and meets its pastel-coloured vertical counterpart at the point $(g, v) = (\hat{G}', 1)$.

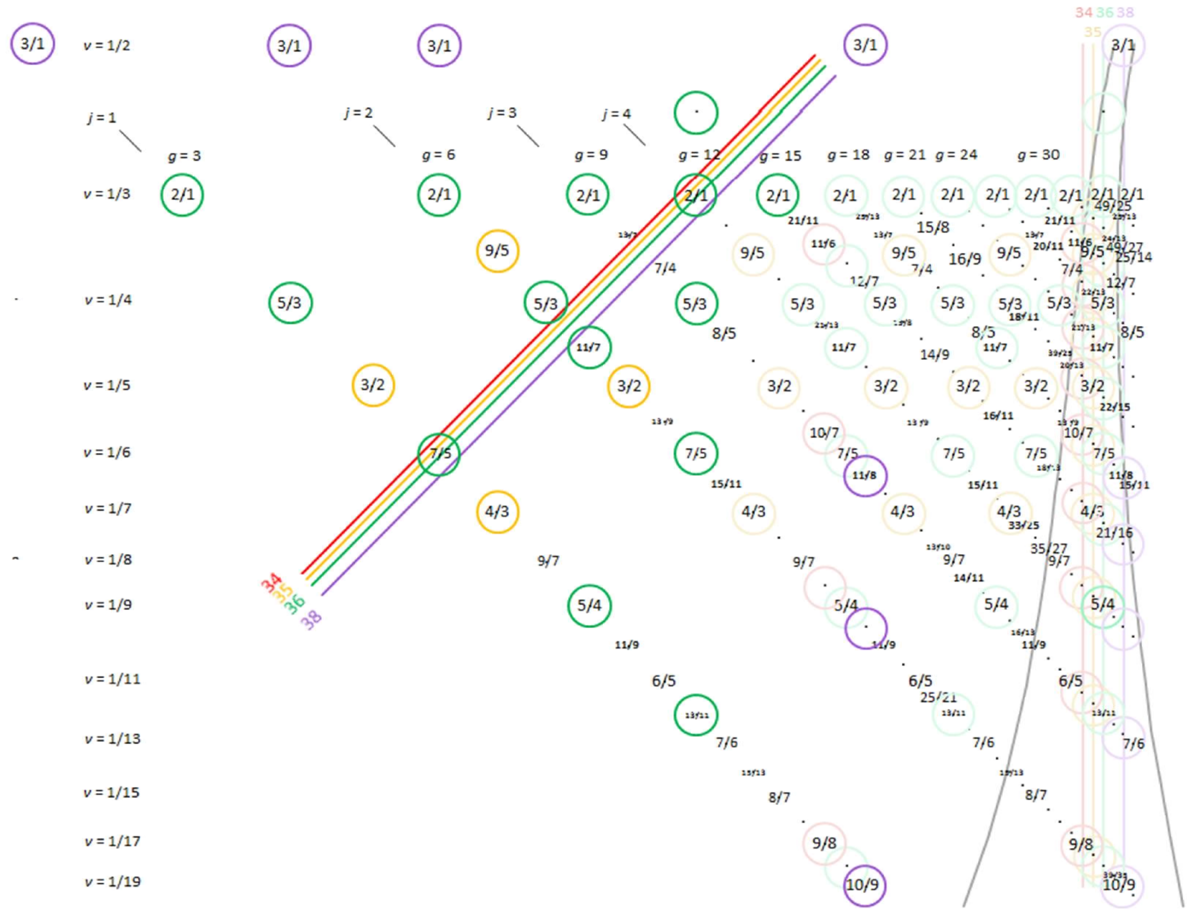


Figure 5. Christmas tree diagram for nearest-step 12edo

Prominent in the diagram is a square of yellow-ringed points representing the intervals f, F and $m_7$. The mirror-pairing (f, F) and the vertical pairing (f, $m_7$) define the Pythagorean and syntonic commas respectively, and these two approximant-matches are sufficient to define 12edo completely in the 5-limit. The green circles are associated with tempered gauge 36. Their abundance is a consequence of this number being rich in factors, as can be seen from the theory presented above.

Christmas tree diagrams for 22edo and 31edo are shown in Figures 6 and 7. In these diagrams only mirror pair matches are shown.

We shall now explore the reasons for the observed characteristics of Christmas tree diagrams, including their symmetries.

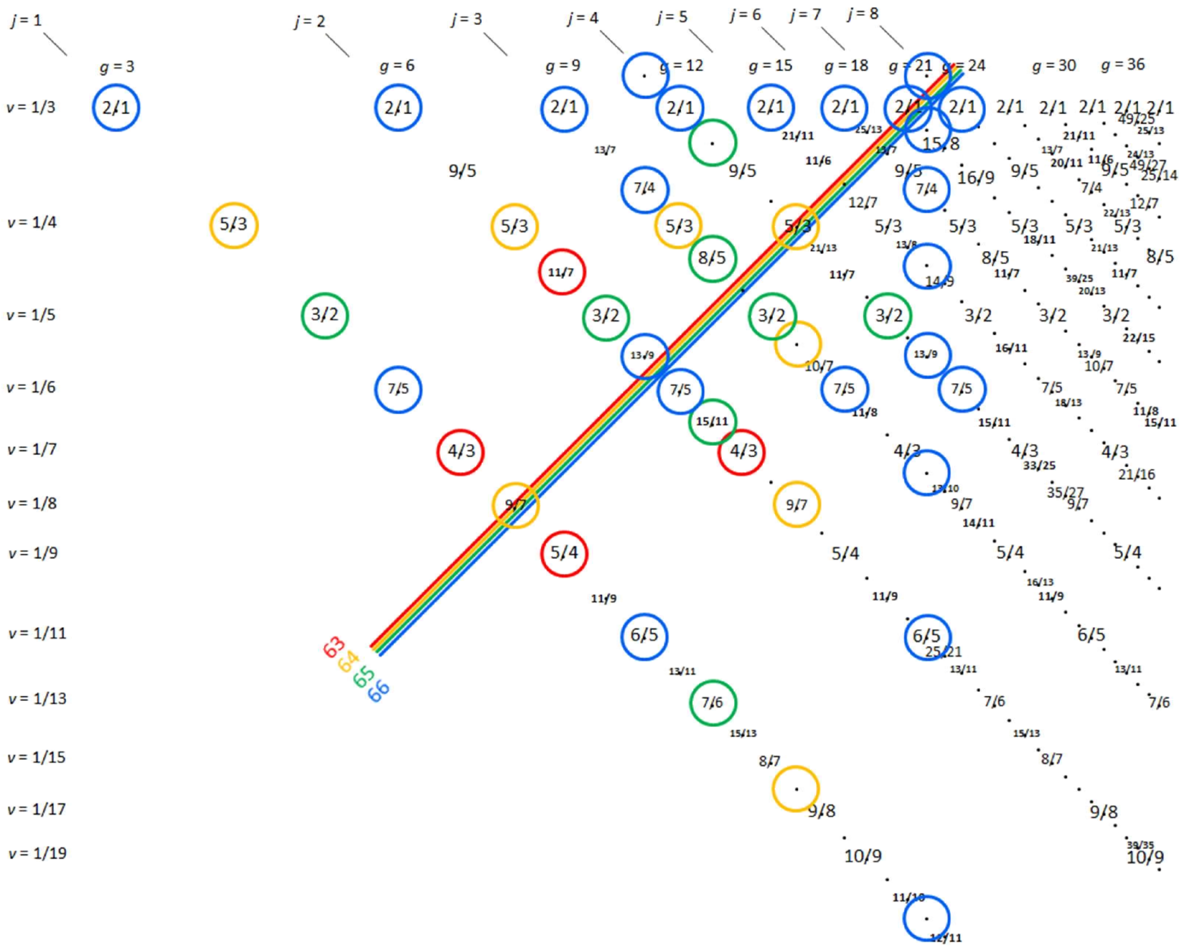


Figure 6. Christmas tree diagram for nearest-step 22edo

If a given ET has matches sharing a tempered gauge $\hat{G}'$ which factorises as $\hat{G}' = F_1 F_2$, then the points

$$(g_1, j_1) = (m_2 F_1, m_1 m_2) \tag{108}$$

$$(g_2, j_2) = (m_1 F_2, m_1 m_2), \tag{109}$$

where $m_1$ and $m_2$ are positive integers, represent intervals belonging to the matched set, provided the points

$$(\hat{G}', \hat{j}_1) \equiv (\hat{G}', g_2) \tag{110}$$

$$(\hat{G}', \hat{j}_2) \equiv (\hat{G}', g_1) \tag{111}$$

lie within the tempered domain. This can be seen by noting that

$$\frac{j_1}{g_1} = \frac{m_1}{F_1} = \frac{g_2}{F_1 F_2} = \frac{\hat{j}_1}{\hat{G}'} \tag{112}$$

and likewise $\frac{j_2}{g_2} = \frac{\hat{j}_2}{\hat{G}'}$ so each of the points represents an interval in the tempered domain.

If $m_2$ is held fixed while $m_1$ ranges over the positive integers, the points $(g_1, v_1)$ are all those which lie on the vertical line defined by $g_1 = m_2 F_1$:

$$(g_1, v_1) = (m_2 F_1, m_1/F_1) \qquad (m_1 = 1, 2, \ldots) \tag{113}$$

and the points $(g_2, v_2)$ are all those which lie on the horizontal line with $v_2 = m_2/F_2$:

$$(g_2, v_2) = (m_1 F_2, m_2/F_2) \qquad (m_1 = 1, 2, \ldots) \tag{114}$$

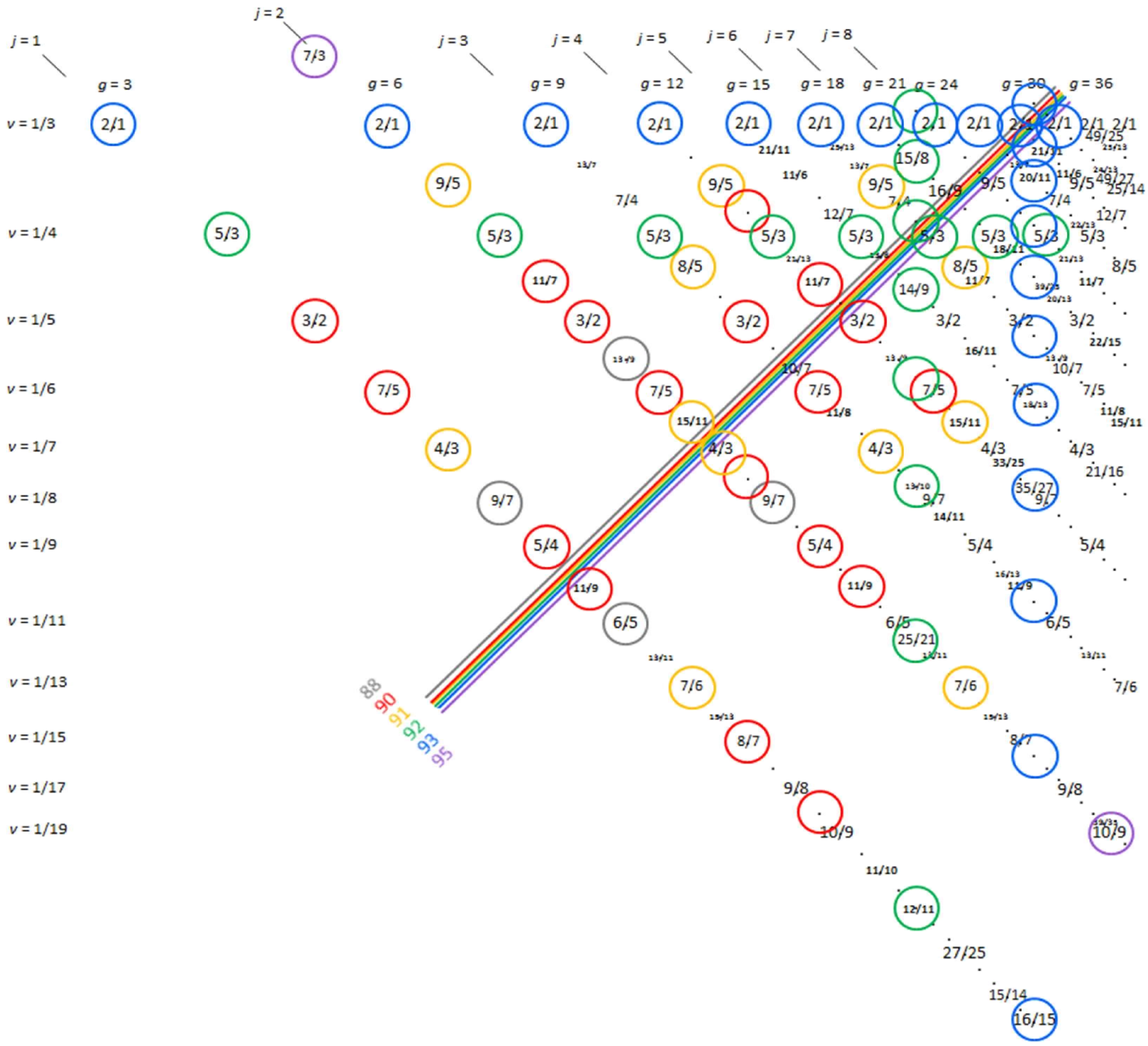


Figure 7. Christmas tree diagram for nearest-step 31edo

For a given value of $m_1$ the points on the horizontal and vertical lines are the reflections of each other in the *mirror line*

$$g = \hat{J}' = \hat{G}'v \tag{115}$$

and thus satisfy

$$g_1 = \hat{G}'v_2 \tag{116}$$

$$g_2 = \hat{G}'v_1 \tag{117}$$

$$j_1 = j_2 = \hat{G}'v_1v_2 = \frac{g_1g_2}{\hat{G}'} \tag{118}$$

We shall refer to such pairs of points as *mirror pairs*. The approximants of the members of a mirror pair share a numerator $j = j_1 = j_2$ (and can thus be termed *diagonal matches*) and their denominators $g_1, g_2$ have product $j\hat{G}'$. In some cases a point may form a mirror pair with itself, in which case we can say that the corresponding interval is *self-matched*.

In summary, given a column of points on the Christmas tree diagram, $(g, v) = \left(\hat{G}', \hat{J}_1'/\hat{G}'\right)$, representing a set of approximant-matched intervals (where $\hat{G}'$ is fixed and $\hat{J}_1'$ takes certain positive integer values) there is, for every factor $F_1$ of $\hat{G}'$, and for every integer multiple $m_2$ of that factor, another column of points $(g_1, v_1) = (m_2F_1, m_1/F_1)$, representing a subset of the original interval set in which $\hat{J}_1' = m_1F_2$ is limited to integer multiples of $F_2 = \hat{G}'/F_1$. Moreover, there is a corresponding row of points obtained by reflecting this column in the line $g = \hat{G}'v$, all these points corresponding to the interval in the matched set represented by the point$(\hat{G}', \hat{J}_2'/\hat{G}')$, where $\hat{J}_2' = g_1$.

Points representing approximant-matched intervals can potentially occur in columns for which $g$ shares a prime factor with $\hat{G}'$. For $g < \hat{G}'$ the number of columns *excluded* by this rule is $\varphi(\hat{G}')$, where $\varphi$ is the Euler totient function.

Mirror pairs, being among the simplest approximant matches to be found in an equal temperament (in the sense of featuring the smallest integers $j$ and $g$), tend to be associated with some of its most distinctive characteristics.

## Interdependence among approximant-matched interval pairs

### Approximant-matched pair triples

A study of approximant-matched interval pairs in a given prime limit for a range of ETs reveals a marked tendency for them to occur in groups of three, with the participating intervals related by addition and subtraction. This observation is formalised in the following theorem.

> If in some equal temperament defined by a val, two interval pairs (*A,X*) and (*B,Y*) are both approximant-matched and have a common difference, *X – A* = *Y – B,* then (*C,Z*) = (*Y–X*, *X*+*B*) is also an approximant-matched pair in that temperament.

Fig 8 is a schematic representation of the interval pairs (*A,X*), (*B,Y*) and (*C,Z*) as vectors (monzos) in tonal space, arranged first as directed edges of a tetrahedron, and second as directed edges of a an octahedron.

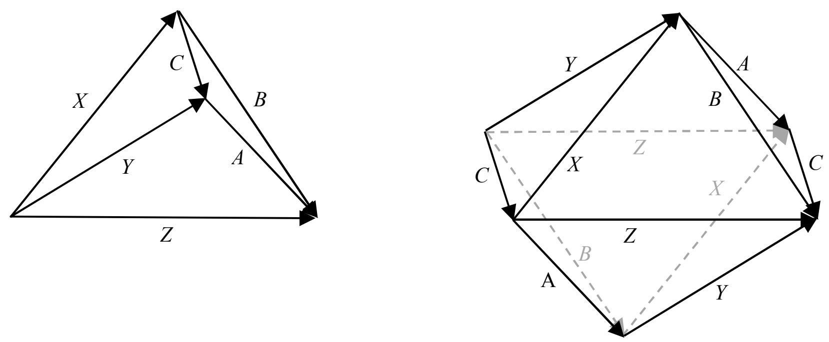


Figure 8. Two representations of an approximant-matched triple in tonal space

The signs of the six intervals have been defined in such a way that it is possible for them all to be positive, and on the diagrams the size of the interval is measured schematically by distance across the page from left to right. In the examples which follow we shall adopt the convention that the intervals *A, B, C, X, Y* and *Z* are positive. This requires that $X < Y < Z$, $X > A$, $Y > B$ and $Z > C$, and as a consequence any six intervals forming an approximant-pair triple can always be assigned unambiguously to these symbols. We shall notate the triple $[(X, A)_{g_A}, (Y, B)_{g_B}, (Z, C)_{g_C}]$, where $g_A$, $g_B$ and $g_C$ are the tempered gauges for the matches in step units. These gauges satisfy $g_A < g_B < g_C$, as can be seen by noting that $g_A$, $g_B$ and

$g_C$ are the tempered gauges of increasing intervals *X*, *Y*, and *Z*, and gauge is an increasing function of interval size.

In the following proof, by contrast, the intervals are unconstrained except for the stated relationship, and in order to highlight the symmetry of the relationships we shall work with the negative of *B*, denoting this by

$$\bar{B} = -B$$

The relationships stated in the theorem can then be expressed (with some redundancy) as:

$$X + C = Z - A = Y \tag{119}$$

$$Y + A = X - \bar{B} = Z \tag{120}$$

$$Z + \bar{B} = Y - C = X \tag{121}$$

$$A + \bar{B} + C = 0 \tag{122}$$

The common sums and/or differences are (from eqns 119-121):

$$X + A = Z - C \tag{123}$$

$$Y + \bar{B} = X - A \tag{124}$$

$$Z + C = Y - \bar{B} \tag{125}$$

From eqns 119-122 it follows that

$$(X + C)(Y + A)(Z + \bar{B}) = XYZ \tag{126}$$

Expanding and substituting using eqns 119-121 again ($C = Y - X$, $A = Z - Y$, $X = Z + \bar{B}$),

$$XYZ + XY\bar{B} + \mathrm{XAZ} + X(Z - Y) + CY(Z + \bar{B}) + (Y - X)AZ + CA\bar{B} = XYZ \tag{127}$$

and simplification followed by division by *XYZ* gives

$$\frac{A}{X} + \frac{\bar{B}}{Y} + \frac{C}{Z} + \frac{A\bar{B}C}{XYZ} = 0 \tag{128}$$

We now show that an identical relationship is satisfied by the approximants of *A*, *B*, *C*, *X*, *Y*, *Z*. Denoting the approximants by lowercase symbols and applying eqn 22 (the summation rule) to eqns 119-122 we find

$$x + c - y - xyc = 0 \tag{129}$$

$$y + a - z - yza = 0 \tag{130}$$

$$z + \bar{b} - x - xz\bar{b} = 0 \tag{131}$$

$$-a - \bar{b} - c - a\bar{b}c = 0 \tag{132}$$

and summing the four equations then gives

$$-xyc - yza - xz\bar{b} - a\bar{b}c = 0 \tag{133}$$

$$\frac{a}{x} + \frac{\bar{b}}{y} + \frac{c}{z} + \frac{a\bar{b}c}{xyz} = 0 \tag{134}$$

Thus, for intervals related in the way described, the approximants satisfy the same relation as the just intervals – somewhat surprisingly considering that eqns 129-132 have extra terms when compared to eqns 119-122.

For a temperament defined by a patent val, the tempered versions of $A$, $B$ and $C$ have the same additive properties as the pure intervals, and therefore satisfy relations of the form 119, 120 and 121. Thus we can write

$$\frac{\hat{A}'}{\hat{X}'}+\frac{\hat{\bar{B}}'}{\hat{Y}'}+\frac{\hat{C}'}{\hat{Z}'}+\frac{\hat{A}'\hat{\bar{B}}'\hat{C}'}{\hat{X}'\hat{Y}'\hat{Z}'}=0 \qquad (135)$$

If $(A,X)$ and $(B,Y)$ are approximant-matched pairs in some temperament, then

$$\frac{\hat{A}'}{\hat{X}'}=\frac{a}{x} \qquad (136)$$

$$\frac{\hat{B}'}{\hat{Y}'}=\frac{b}{y} \qquad (137)$$

and to satisfy 134 and 135 it follows that the pair $(C,Z)$ must also be matched:

$$\frac{\hat{C}'}{\hat{Z}'}=\frac{c}{z} \qquad (138)$$

Since the proof is indifferent to the sign of the intervals, it follows that whenever an equal temperament defined by a patent val has two approximant-matched interval pairs for which the sum or difference of one pair is equal to the sum or difference of the other pair, the temperament will also have a third matched pair.

**Mirror pair property of intervals featuring in triples**

Another notable feature of triples is that their constituent approximant-matched intervals are mirror pairs.

To understand the reason for this we need the following relationship between the approximants $a$, $b$, $c$, $x$, $y$, $z$, which can be proved using a procedure similar to that used to derive eqn 134:

$$ax-by+cz-ax\bar{b}ycz=0 \qquad (139)$$

Using the notation

$$\hat{G}'_{AX}\equiv\frac{\hat{A}'}{a}=\frac{\hat{X}'}{x} \qquad (140)$$

$$\hat{G}'_{BY}\equiv\frac{\hat{B}'}{b}=\frac{\hat{Y}'}{y} \qquad (141)$$

$$\hat{G}'_{CZ}\equiv\frac{\hat{C}'}{c}=\frac{\hat{Z}'}{z} \qquad (142)$$

and in expectation of a result concerning mirror pairs, we define

$$j_{AX}=\hat{G}'_{AX}ax=\hat{A}'\hat{X}'/\hat{G}'_{AX} \qquad (143)$$

$$j_{BY}=\hat{G}'_{BY}by=\hat{B}'\hat{Y}'/\hat{G}'_{BY} \qquad (144)$$

$$j_{CZ}=\hat{G}'_{CZ}cz=\hat{C}'\hat{Z}'/\hat{G}'_{CZ} \qquad (145)$$

Comparing eqns 143 and 118 it is evident that $(A,X)$ is a mirror pair if and only if $j_{AX}$is an integer (with analogous statements applying to the other pairs).

Combining eqns 143-145 with 139 we find

$$\frac{j_{AX}}{\hat{G}'_{AX}}-\frac{j_{BY}}{\hat{G}'_{BY}}+\frac{j_{CZ}}{\hat{G}'_{CZ}}-\frac{j_{AX}j_{BY}j_{CZ}}{\hat{G}'_{AX}\hat{G}'_{BY}\hat{G}'_{CZ}}=0 \qquad (146)$$

Applying eqn 22 to this relationship (treating $j_{AX}/\hat{G}'_{AX}$ etc. as approximants) we deduce that the ratios taking the place of the corresponding frequency ratios must satisfy

$$\frac{(\hat{G}'_{AX} + j_{AX})(\hat{G}'_{BY} - j_{BY})(\hat{G}'_{CZ} + j_{CZ})}{(\hat{G}'_{AX} - j_{AX})(\hat{G}'_{BY} + j_{BY})(\hat{G}'_{CZ} - j_{CZ})} = 1 \tag{147}$$

Barring improbable coincidences, this can only be true if there is wholesale cancellation, which means, given that $\hat{G}'_{AX} < \hat{G}'_{BY} < \hat{G}'_{CZ}$ and the $j$ terms are positive, the numerator and denominator terms must equate pairwise as follows:

$$\hat{G}'_{AX} - j_{AX} = \hat{G}'_{BY} - j_{BY} \tag{148}$$

$$\hat{G}'_{BY} + j_{BY} = \hat{G}'_{CZ} + j_{CZ} \tag{149}$$

$$\hat{G}'_{CZ} - j_{CZ} = \hat{G}'_{AX} + j_{AX} \tag{150}$$

This implies

$$j_{AX} = \hat{G}'_{CZ} - \hat{G}'_{BY} \tag{151}$$

$$j_{CZ} = \hat{G}'_{BY} - \hat{G}'_{AX} \tag{152}$$

$$j_{BY} = j_{AX} + j_{CZ} \tag{153}$$

which, for integer $\hat{G}'_{AX}$, $\hat{G}'_{BY}$ and $\hat{G}'_{CZ}$, means that $j_{AX}$, $j_{BY}$ and $j_{CZ}$ are integers, and consequently (barring coincidences) that ($A$,$X$), ($B$,$Y$) and ($C$,$Z$) are mirror pairs.

***Examples: 5-limit triples***

The following are some 5-limit triples featuring in well-known ETs, some of which can be found in Figures 5, 6 and 7.

| | | |
|---|---|---|
| 7edo: | $[(F, M_6)_{20}\,(m_7, o)_{21}\,(m, P_{12})_{22}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (1, 2, 1)$ |
| 12edo: | $[(F, m_7)_{35}(M_6, o)_{36}\,(t, P_{12})_{38}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (2, 3, 1)$ |
| 19edo: | $[(m, F)_{55}\,(f, M_6)_{56}\,(t, o)_{57}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (1, 2, 1)$ |
| 22edo: | $[(M, f)_{63}\,(F, m_6)_{65}\,(m, o)_{66}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (1, 3, 2)$ |
| 26edo: | $[(t, M_6)_{76}\,(m, m_7)_{77}\,(s^+, o)_{78}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (1, 2, 1)$ |
| 31edo: | $[(M, F)_{90}\,(f, m_6)_{91}\,(s, o)_{93}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (2, 3, 1)$ |
| | $[(m_6, m_7)_{91}\,(M_6, M_7)_{92}\,(X, P_{12})_{98}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (6, 7, 1)$ |
| 53edo: | $[(T, M)_{153}\,(m, f)_{154}\,(s, F)_{155}]$ | $(j_{AX}, j_{BY}, j_{CZ}) = (1, 2, 1)$ |

The12edo triple in this list is illustrated in Figure 9. It can be regarded as a consequence of the common difference (a minor third) between the first two matched pairs. The consequential match $(C,Z) = (t,P_{12})_{38}$ is typical of matches derived this way, pairing a small interval with a large interval. Approximant-matching involving a large interval tends to produce a large fractional error, but when applied to the small interval the result is an error which is tolerable in absolute terms.

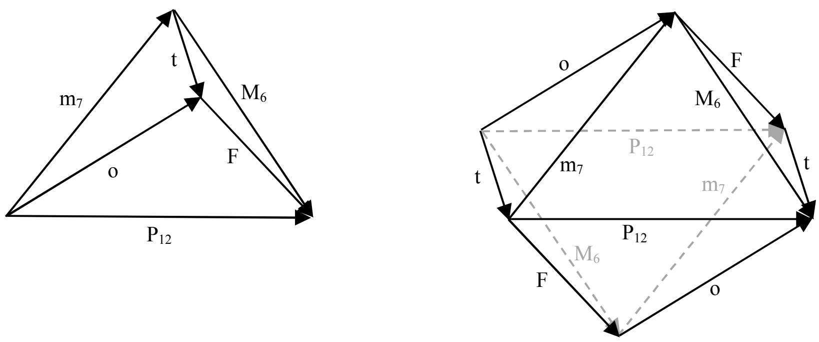


Figure 9. 5-limit 12edo triple

**Miscellaneous results**

Some further results applying to approximant-matched pair triples under the assumptions used above are recorded here for completeness.

Equations of the form 128 may be recast in a variety of ways. For instance, after multiplying through by a factor $(X/A)(Z/C)$, eqn 128 becomes

$$\frac{X}{A}+\frac{\bar{B}}{Y}+\frac{Z}{C}+\frac{X\bar{B}Z}{AYC}=0 \qquad (154)$$

By manipulating eqns 119-121 and their counterparts for tempered intervals it can be shown that

$$AX-BY+CZ=0 \qquad (155)$$

$$\hat{A}'\hat{X}'-\hat{B}'\hat{Y}'+\hat{C}'\hat{Z}'=0 \qquad (156)$$

It then follows from eqns 143-145 that

$$j_{AX}\hat{G}'_{AX}-j_{BY}\hat{G}'_{BY}+j_{CZ}\hat{G}'_{CZ}=0 \qquad (157)$$

The vals of the six intervals comprising a triple can be expressed in terms of exactly three linearly independent basis vals. This is because the integers $\hat{A}'$, $\hat{B}'$ and $\hat{C}'$ are sufficient to define both the triple and three components of the temperament's val. If the val had other than three independent components it would be under or over-specified by this procedure. For this reason there can be no 'Pythagorean triples' in our sense of the term. There is nothing to prevent the intervals comprising a triple featuring more than three prime factors, however.

Finally we note that the relationship between $J$ and $v$ (eqn 9) is identical to that between rapidity and velocity (relative to light speed) in special relativity. This allows another interpretation (of uncertain significance) that can be attached to a result derived above: for four observers moving at relativistic speeds along a straight line, a relationship which is satisfied by certain velocity ratios (eqn 128) is also satisfied by the corresponding rapidity ratios (eqn 134).

## Quartets of approximant-matched pair triples

Certain ETs have been found to contain examples of a more complex structure formed from six approximant-matched interval pairs participating in four interconnected triples. This structure can be understood by means of the following theorem.

> If in some equal temperament defined by a val, three approximant-matched interval pairs (*A*,*X*), (*B*,*Y*) and (*D*,*U*) (where each interval may be either positive or negative) satisfy $X-A=Y-B=U-D$, the temperament contains four interconnected triples built from combinations of six approximant-matched interval pairs.

The proof is as follows. Using the preceding results we can immediately identify three triples which we shall notate thus:

{(*A*,*X*) (*B*,*Y*) (*C*,*Z*)} {(*A*,*X*) (*D*,*U*) (*E*,*V*)} {(*B*,*Y*) (*D*,*U*) (*F*,*W*)}

(where in this case we drop the tempered gauge subscript and relax the constraints relating to signs and relative magnitudes which normally apply with this notation).

To prove the existence of a fourth triple we view eqn 134 as a relation between approximants *a*/*x*, *b*/*y* and *c*/*z*, and transform these approximants to frequency ratios $r_{a/x}$, $r_{b/y}$ and $r_{c/z}$, where

$$r_{a/x}=\frac{x+a}{x-a}, \qquad r_{b/y}=\frac{y+b}{y-b}, \qquad r_{z/c}=\frac{z+c}{z-c} \qquad (158)$$

Since the approximants may in these cases have magnitude greater than one, the frequency ratios may be negative. To avoid the inconvenience of dealing with complex intervals we shall work with the ratios and approximants only, for which eqns 21 and 22 remain valid. Applying eqn 22 to eqn 134 and its counterparts for the two other triples yields

$$r_{a/x}\, r_{b/y}^{-1}\, r_{c/z} = 1 \tag{159}$$

$$r_{a/x}^{-1}\, r_{d/u}\, r_{e/v}^{-1} = 1 \tag{160}$$

$$r_{b/y}\, r_{d/u}^{-1}\, r_{f/w} = 1 \tag{161}$$

Combining these three equations by multiplication we then obtain

$$r_{c/z}\, r_{e/v}^{-1}\, r_{f/w} = 1 \tag{162}$$

and converting back to the approximant domain yields

$$\frac{c}{z} - \frac{e}{v} + \frac{f}{w} - \frac{cef}{zvw} = 0 \tag{163}$$

By applying an identical process to the counterparts of eqn 135 on the assumption of consistency we obtain

$$\frac{C'}{Z'} - \frac{E'}{V'} + \frac{F'}{W'} - \frac{C'E'F'}{Z'V'W'} = 0 \tag{164}$$

Since (*C*,*Z*) are (*E*,*V*) are matched, it then follows that (*F*,*W*) is also matched, so a fourth triple {(*C*,*Z*), (*E*,*V*), (*F*,*W*)} is established.

***Example: quartet of triples in 31edo***

A quartet of 7-limit approximant-matched triples in 31edo is shown in Figure 10 In this case all the featured intervals, if tuned to the nearest step, are consistent with the patent val.

In this figure, vertices of a tetrahedron represent triples featuring approximant-matched pairs identified with the adjacent edges. The quartet of triples can be set out as follows:

{ [(m, 9/7)$_{88}$ (F, 7/5)$_{90}$ (7/6, m$_7$)$_{91}$] [(m, 9/7)$_{88}$ (M$_6$, 14/9)$_{92}$ (35/27, o)$_{93}$]
[(F, 7/5)$_{90}$ (M$_6$, 14/9)$_{92}$ (t, 7/3)$_{95}$] [(7/6, m$_7$)$_{91}$ (35/27, o)$_{93}$ (t, 7/3)$_{95}$] }

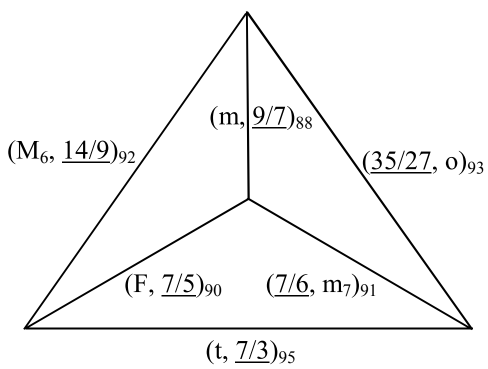


Figure 10. 7-limit quartet of triples in 31edo

***Example: quartet of triples in 39edo***

Figure 11 shows a quartet of 5-limit approximant-matched triples in patent 39edo.

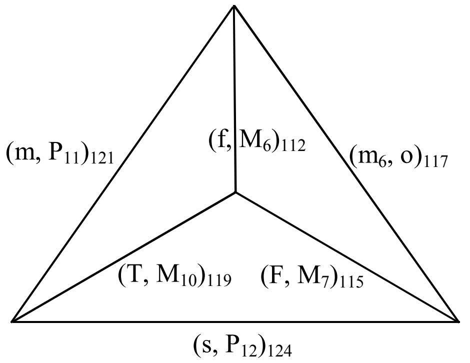


Figure 11. 5-limit quartet of triples in patent 39edo

***Example: quartet of triples in 67edo***

Figure 12 shows a quartet of 5-limit approximant-matched triples in patent 67edo.

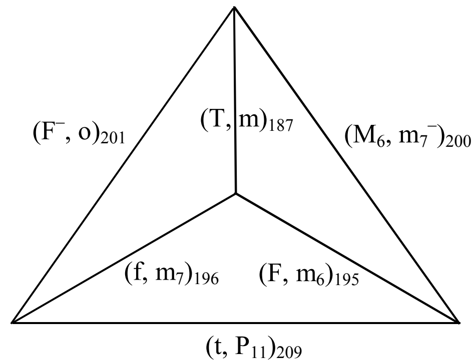


Figure 12. 5-limit quartet of triples in patent 67edo

## Conclusions

The bimodular approximation expresses the sizes of just intervals as simple fractions which we have termed *bimodular approximants*. While this rational approximation to the logarithm function has a long history as a method for estimating the sizes of small musical intervals, its relevance to larger intervals and equal temperaments has not previously been fully appreciated. In this context it has an explanatory power which arguably provides a more direct route to understanding certain aspects of equal temperaments (including equal divisions of non-octave intervals) than the frequently-aired type of explanation based on continued fractions. Closely related to this is its ability to provide a rationale for certain seemingly fortuitous near-rational relationships between intervals which have long been known to theorists. *Bimodular commas* express the degree to which such approximant-derived relationships depart from exactness, and commas of this type are always present among those which an equal temperament shrinks to zero (tempers out).

A space in which frequency difference is plotted against frequency sum provides a convenient framework for graphical presentations of the theory. In this space, just intervals are represented by points with integer coordinates, and columns of such points form sequences of approximately equally spaced intervals (*near-equal progressions*) that require only slight adjustment to bring them into coincidence with equal temperaments. A region of the space enclosed by curved lines (the *Jacana diagram*) defines the precise relationship between just intervals and their nearest-step tunings in a specific equal temperament, and a variant of this diagram plotted on logarithmic axes (the *Christmas tree diagram*) highlights relationships between pairs of low-complexity intervals which are associated with distinctive characteristics of the temperament. Examples have also been found of more complex sets of relationships linking bimodular approximants with equal temperaments.

This work is being extended to an investigation of other types of logarithmic approximation which show promise as tools for the study of tuning systems.

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