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PRINCE CHU TSAI-YÜ'S LIFE AND WORK

A RE-EVALUATION OF HIS CONTRIBUTION TO EQUAL TEMPERAMENT THEORY*

Fritz A. Kuttner

In Western accounts of Chinese musical history Chu Tsai-yü[1] is one of the most frequently mentioned personalities, but unfortunately very few facts have been accurately reported about this eminent musical historian and theorist who flourished toward the end of the 16th century. Since so far there exist no published translations in Western languages of his many writings, it seems desirable to present the most important data of his work and life, particularly because some of his achievements may have had a profound and lasting influence on certain musical developments in Europe. For Chu Tsai-yü has frequently been credited by respectable authors with what is usually called "the invention of equal temperament." As we shall see, this claim cannot be considered correct without major qualifications, depending on the point of view one takes. It has also been stated with somewhat emotional emphasis that the Prince's contribution to music theory represents "the crowning achievement of China's two millennia of acoustical experiment and research" (Robinson 1962:224). Since the achievements of these two millennia, as we have come to realize in the last twenty years, have been magnificent in many respects, I must express doubts as to the justification of that second claim. More justified, I believe, would be the conclusion that Chu was one of the most important historians of his nation's music.

Chu was born in 1536 in Ho-nei, Honan Province, into a family of high aristocratic standing. He was a descendant, in the sixth generation, of Hung Hsi, the fourth emperor of the Ming Dynasty (1368-1644). His father Chu Hou-huan was hereditary Prince of Chêng. Hence, in Western literary sources our scholar is usually referred to as Prince Chu (a title which should be rendered in German as *Fürst,* not as *Prinz*). As will be seen below, it is questionable whether the princely title should always be added to the family name.

*This paper had originally been commissioned and written in 1968 for a planned Festschrift in honor of Professor Marius Schneider, Cologne University. The manuscript was typeset and proofread in 1969 but never published because the project ran out of funds before going to press. What follows here is a completely reworked and enlarged 1974 version of the previous article. (Ed. Note)

Chu's father was a learned man of high principles, deeply devoted to the teachings of Confucian tradition which demand humility, filial piety and obedience from those committed, in addition to an austere and puritanical style of life. These views clashed with the life and attitudes at the imperial court and induced Chu Hou-huan in 1548 to submit a memorial to Emperor Chia Ching (1522-1567) in which he criticized the sovereign and urged reforms, particularly with regard to the court's strong preference for Taoist over Confucian rituals. This step proved disastrous for the family. Although the emperor did not react immediately to the provocation, he did not have to wait long for an opportunity to get even with his clansman. Two years later a cousin of the older Chu, with some grudge of his own on his mind, aware of the emperor's sentiment and hopeful to gain the princely rank and title for himself, officially accused Chu of treason and many other alleged violations. The ad-hoc imperial commission charged with investigating the case dismissed all accusations except for some trivial offenses, but their findings were sufficient for the emperor to deprive the prince of his rank and to sentence him to confinement in a prison reserved for convicted members of the imperial clan. Chu Hou-huan served more than 19 years in this "Festungshaft."

When this happened (1550), the younger Chu was 14 years old. Embittered by his father's severe and unjust punishment, he left his family's ancestral residence in protest and lived alone in a small cottage outside the palace gate. There he devoted himself to the same studies his father had pursued: mathematics, calendrical science and astronomy, musical theory and acoustics.[2]

In 1567, Emperor Lung Ch'ing succeded Chia Ching on the throne, and two years later he granted amnesty to Chu and reinstated his rank and title. Only then (1570) did his son marry, at the age of 34, a most unusual restraint in a nation where men of social standing customarily married at 19 or even earlier. Filial piety and the Confucian concept that a son should not enjoy the fruits of marital life while his father is deprived and in misery, were, of course, the reason for the younger Chu's traditional conduct.

It seems that for the next 21 years Chu Tsai-yü gave most of his time to assisting his father in his scholarly work, since only two works (see items 1 and 4 of the selected works in appendix) of modest significance were published during that period, in 1581 and 1584. The conclusion that Chu may have matured rather late as a scholar and writer seems unconvincing; again I must assume that filial subordination, obedience and restraint by Confucian principles forbade any obvious and successful achievement during the father's lifetime. These delaying circumstances were aggravated by the fact that father and son were active in the same scholarly disciplines, and that the elder had been his son's most important teacher. The filial liberation from restraint came in 1591 when Chu Hou-huan passed away. By then Chu Tsai-yü was 55 years old, and in fast succession he completed one important work after the other.

This has caused justified speculation about the possibility that many of these works may have been completed earlier, but kept in the drawer for later publication after the father's demise. Since such speculations play a role in a context that will occupy us presently, the following considerations should be kept in mind: (1) There is no factual evidence available so far to support the earlier dating of certain important works, except the mentioned circumstances which make the idea plausible but not provable. (2) A significant factor speaks against an earlier dating proposition: the one work which Chu believed to be his greatest and which deserves more than usual interest in Western musicological studies (item 6 of appendix), was not completed until 1595 or 1596. All the other works came out between 1601 and 1606, while in the four years following his father's death only one work was published: in 1595 he submitted to the court a collection of three treatises on proposed calendar reforms. And although quite a few of the works published in 1606 were unquestionably completed earlier and kept waiting for a collective edition of 1606, there is not a shred of evidence that any of these writings were ready for publication before 1591, or even before 1596. This is not the situation of a man whose drawers are full of completed manuscripts, and who, in addition, possesses rank and financial means for publishing anything he wants to, at the earliest possible occasion.

In 1593, at the end of the traditional two-years' mourning, Chu was entitled to assume the hereditary title and rank of Prince of Chêng. But he resigned his title and petitioned the court of Emperor Wang-hi (1573-1620) to grant his abandoned rank to—of all persons!—his father's cousin, the same man whose accusations had caused his late father's ordeal and imprisonment. The emperor, most reasonably, hesitated for 13 years to grant this request, and only in 1606, after repeated appeals to the throne, was the title of "Seventh Prince of Chêng" given to Chu's relative. Since it was in the same year that Chu submitted to the court his large collective work of ten separate musicological treatises, thus establishing at one stroke his scholarly reputation and scientific merit, the emperor created, in recognition of such exceptional merit and in a special decree, a new rank of heir-apparent to a prince of the first degree. Thus, under the Chinese custom of posthumous names for distinguished persons, Chu Tsai-yü was known after his death as Chêng Tuan-Ch'ing Shih-tzu (= hereditary prince), Tuan-Ch'ing being the posthumous name. This established a princely title for him as of 1606, and up to his death in 1611; for the first 70 years of his life he did not have that title, partly because he did not inherit it until 1593, and partly because he had renounced it between 1593 and 1606.

Chu's resignation of rank and his appeal to have it transferred to that questionable cousin has been praised as an unprecedented act of humility, generosity and self-denial. This it may well be. But I cannot help feeling that it was also a unique, lifelong case of carrying a "chip on the shoulder," materializing 43 years after his father's imprisonment, 24 years after his father's liberation and rehabilitation, and carried on for more than half a century, from Chu's 14th to his 70th year of age. The Western observer struggles in vain to fathom the most profound depths of the Confucian-trained mind and to suppress the notion of a terrible distortion of the filial piety postulate which attempts to reward abominable meanness and greed with the highest honor of social rank his nation had to bestow.

Prince Chu died on May 19, 1611 in the district of Ho-nei, Honan province.[3]

His works span a wide range of topical contents. The mathematical writings deal with geometrical subjects and the solid geometry of dry and liquid measure, including a survey of all measurement standards in Chinese history. The works on calendrical science were stimulated by inaccuracies in the calendar system of the Ming dynasty which led repeatedly to wrong calculations for eclipses and other celestial phenomena. These mathematical and astronomical studies had been started by his father after whose death Chu continued and enlarged them. The *Li Shu* (No. 2), submitted to the court in 1595 and printed a few years later did not, however, improve the shortcomings of the Ming calendar because it was still based on the inadequate method of earlier reform attempts which set the length of the solar year at 365¼ days. The use of that figure causes an error of ca. 10 min., 14 sec. per year, or nearly a full day for every 128 years. This work, therefore, must be considered a scientific failure. That its recommendations were not adopted by the imperial astronomical commission, an ultra-conservative body, was probably based on political reasons rather than on an awareness of its mathematical weaknesses: the commission resisted in principle any calendar reform.

The greatest significance of Chu Tsai-yü's work, then, lies unquestionably in his achievements in the musicological field where he covered many historical aspects of music and dance, especially those of antiquity, and in the theory or acoustics of the Chinese tone system. Three works are of particular interest to us because of their implications for Western musical theory:

a–*Lü Hsüeh Hsin Shuo,* published 1584 (No. 4);
b–*Lü Lü Ching I,* written 1595/96, but probably not published before 1606 (No. 6);
c–*Suan Hsüeh Hsin Shuo,* published 1603 (No. 3).

All three titles contain substantial contributions to the numerical definition of equal temperament, partly in monochord tabulations of amazing precision, partly in highly detailed mathematical calculations. The earliest treatise (No. 4) contains a complete nine-digit monochord[4] of equal temperament with the octave ratio 10:5 for what appears to be a calculation of string lengths; for the lengths of pitch-pipes there are monochords based on the octave ratios 100:50 and 90:45, with four decimals, as well as tabulations of pipe diameters and circumferences on the base 100:50 with two decimals.

The next work (No. 6) completed eleven or twelve years later, contains an enormously detailed mathematical investigation of all conceivable parameters involved in the definition of pitches in equal temperament tuning, including string lengths and pitch-pipe dimensions, such as tube lengths, inner and outer diameters, circular surface areas, bore and volume of pipes through three octave ranges in 9- and 10-digit tabulations. There are even "slanted" (i.e. diagonal or hypothenuse-like) sections for all 36 pipes. Some of the computations are executed for a variety of measuring standards traditionally used in Chinese musical history, such as the numbers 9 and 81 as base units for the starting tone *Huang Chung*, the Chinese equivalent of C or Do. Other tabulations of the monochord values are calculated with the octave ratios 2:1 or 10:5.

Throughout the work *Lü Lü Ching I* the twelfth root of 2 is numerically implied as the quantitative definition of the semitone in equal temperament, but it is never stated explicitly as a mathematical expression.[5] (Appendix No. 1 discusses the contents of this treatise in more detail.)

The last of the above three works (No. 3) contains the string (or pipe?) lengths for 36 pitches through three octaves with 22(!) decimals in equal temperament, besides giving pitch-pipe diameters for the lowest and middle of the three traditional Chinese octaves.

Thus, Chu's presentation of equal temperament for his nation's tone system is a careful, thorough and comprehensive one, based on solid mathematical and arithmetical knowledge. Although he refers frequently to previous, related achievements in the history of Chinese musical theory—some of them going allegedly back as far as the second century B.C.—there cannot be the slightest doubt that we are here confronted with original and independent findings of considerable theoretical importance. The Prince himself took great pride in his work, considered it as the most significant accomplishment of his scholarly career to date, and stated in the preface to the *Lü Lü Ching I* that he was publishing findings never known and reported before.

Controversy arises in this context because of an amazing coincidence in musical history. At roughly the same time as Chu completed his above three works, Simon Stevin (1548-1620), a distinguished Flemish mathematician and inventor, drafted—with no particular care or urgency—an essay containing the mathematical formulation of equal temperament as $\sqrt[12]{2}$ for the first time in Western musical theory; he continued with the calculation of a monochord which defines the 12 semitone values, correct to four decimal places, as the 12 successive powers of the twelfth root of 2. He then sent the manuscript to a scholarly friend where it eventually got misplaced or forgotten. Obviously Stevin was not much interested in a publication of his findings because he did not attach too much significance to them. The essay remained forgotten, again apparently, because the recipient as well did not consider it as something of real scientific consequence. Thus, it took until 1884, when it was rediscovered and edited by de Haan, to be published for the first time (Stevin 1884).

This close coincidence has given rise to arguments in which the respective authors try to prove one or several of the following hypotheses:

1–that Prince Chu has unquestionable and unqualified priority of this "invention";
2–that Stevin's formulation comes later than Chu's by anywhere from 1 year (1584-1585) to 24 years (1584-1608);
3–that Stevin's "duplication" in Holland was not an independent act of research or reasoning, but based on unspecified information reaching him from China via hypothetical messages or messengers;
4–that Stevin did not give credit to Chu and may therefore be guilty of plagiarism. (The fact that Stevin never made any attempt to publish his results remains conveniently ignored).

Depending on which side is being taken, the argument is sometimes colored by emotional components of Sinophilism or, on the other hand, by the desire to demonstrate a superiority of the Occidental scientific mind over that of the Far East. Such preoccupation with preset objectives may play all kinds of tricks on the authors involved, from negligence in determining the scientific and musical situations existing at that time on both continents, to "correcting" important time or other factors on which the establishment of a priority depends.

In Western literature Robinson's magnificent essay on sound (acoustics) in China makes an elaborate plea in favor of Prince Chu's unqualified priority. The important locus of this publication (Robinson 1962:126-228) and the great prestige that the author's reasoning derives from its incorporation in Sir Joseph Needham's monumental work has, in Western sinological and musicological circles, created a onesided impression of the priority question which is in need of a re-evaluation.[6]

Any investigation into the priority of an invention has to consider more than mere chronology. In particular, the question must be answered whether conflicting priority claims involve identical inventions. Furthermore, a responsible decision calls for a careful comparison between the different scientific and cultural environments in which the priority contestants worked and created; the respective "state of the art" in different environments could play a crucial role in our final judgement. Chronologically, there is no doubt that Prince Chu was the first to offer, in 1584, a nine-digit monochord of the 12 pitches of equal temperament. As we shall see in a moment, Stevin's monochord and the description of his calculation method came later. But Chu's 1584 presentation does not contain a mathematical or theoretical definition of the temperament; it is strictly a numerical or figuring exercise the procedure of which is only partly indicated by the Prince. Stevin's later presentation, however, defines the temperament as a series of mean proportionals between two extremes, calculates the semitone as the twelfth root of 2, and the 12 monochord pitches as the 12 consecutive powers of that twelfth root.

So far, all authors writing on Prince Chu's achievement, with the exception of Robinson, confess that they do not know how he calculated his nine-digit monochord. Appendix No. 1 presents a demonstration of the calculation procedures used by Chu in 1584 and in 1595/96, and a discussion of Robinson's interpretation of this topic; we shall then find that his information is partly correct in one direct quotation from Chu's 1584 treatise, and partly wrong as concerns an erroneous speculation.

The Stevin manuscript is said by several authors, among them Barbour (1951:76), to date from 1595 or 1596, but without any evidence in support of this assumption. De Haan's edition, to the best of my knowledge at this time the only authoritative source, refrains from any dating proposition and does not even consider the time of the presumable completion of Stevin's treatise. The appendix A to de Haan's edition contains a letter to Stevin written by the organist Abraham Verheijen of Nijmegen, which is apparently part of a correspondence between the two men about the contents of Stevin's manuscript (Stevin 1884:87 ff). One gets the impression that Stevin wanted to hear the opinion of a practical, experienced musician, and that they were discussing the respective merits of meantone versus equal temperament, with Stevin advocating the latter tuning system. Unfortunately, this letter is undated and provides no information as to the time Verheijen studied the Stevin manuscript.

Other speculations about the presumable date of origin might include the following considerations. In his treatise Stevin refers to his work on arithmetics where he gives, in proposition No. 45, a method of calculating a number of mean proportionals between two given figures.[7] He also defines the problem of equal temperament as the construction of eleven mean proportionals between the numbers 1 and 2 which represent the higher and lower octaves. The resulting formula $\sqrt[12]{2}$, then, gives the size of an equally tempered semitone, and each interval between the lower octave and one of the 12 tempered semitones is represented by one of the 12 powers of that formula. Now, the arithmetical work was published in French in Leyden in

1585, and one must assume that the manuscript on temperament should have been written in 1585 or later.[8] How much later is open to any guess.

A rather vague clue has been provided by Depau who pointed out that Stevin stopped any work of a purely abstract or scientific nature after 1586:

> On peut dire que Stevin ne fait plus de sciences pures après 1586. De 1586 à 1605, l'une de ses activités dominantes semble être la résolution de problèmes techniques (Depau 1942:112).

Since Depau's statement was made outside any context of possible controversy about a priority question and since, obviously, he had no knowledge of Prince Chu's achievement in the Far East, we may consider him as neutral and unbiased. But that is hardly good enough to support a conclusion that Stevin's manuscript must have been completed by the end of 1586. Equally inconclusive is Robinson's suggestion that *Spiegeling der Singconst* may have been written as late as 1605-1608 (Robinson 1962:228); there is absolutely nothing to support such arbitrary dating. Whether we like it or not, the treatise may have been written in 1585 or 30 years later, for all we know so far.

Another argument, however, has more merit. The formulation of a twelfth root of 2 did not present particular difficulties to mathematicians in Europe or in China at the end of the 16th century. In fact, the formula is quite simple once it is realized what the proportional quantities involved in the equal temperament problem actually are. Several European theorists writing before Stevin and Prince Chu had defined it as the construction of eleven mean proportionals between the numbers 1 and 2. The difficulty was in the arithmetic calculation of these proportionals, and specifically in the extraction of a twelfth root. How this was to be solved, Stevin had shown in his *Arithmètique,*[9] and thus one could state with some justification that the arithmetical formulation of equal temperament and its method of calculation had been presented by Stevin in 1585. The day he actually calculated the monochord derived from his formulation—in 1585 or much later—does not seem to matter much; the method had been determined and the figuring task was no more than mechanics. On the basis of this reasoning we are inclined to allot to Stevin a partial and limited priority, not of the "invention," but of the arithmetical definition and of one corresponding method of calculation; we are also inclined to state the date of his solution as 1585. The date that should be credited to Prince Chu for the calculation of his first monochord was stated above as 1584.

Apart from the somewhat sterile search for priority dates and circumstances, we emphasize our conviction that unquestionably Stevin and Chu worked in complete independence from each other, without knowledge of the work done on the other side of the globe. They both produced original thought and results and have both claim to independently achieved solutions. For a full understanding of the question of "invention" (rather than arithmetical definition and calculation methods), however, we must consider the circumstances existing in temperament theory and practice before and around 1580/1590 in Europe and China.

HISTORICAL DEVELOPMENTS

The history of temperaments reaches back into the second or even third centuries B.C., both in East Asia and in Europe. In its essence, tempering is a quantitative problem, and countless solutions have been offered by numerous theorists or musicians, many of them proposing workable and sometimes satisfactory approximations to certain "ideal" but irrational quantities. Since every temperament system is a compromise between conflicting aesthetical, traditional or practical postulates, there cannot exist any single "correct" or "definitive" mathematical solution, nor is there any absolute or final precision in quantitative results whose usefulness is determined by their practical applicability. Thus, Stevin's four-digit monochord is "less precise" than Chu's nine-digit calculation, but a fifty-digit tabulation, easily printed out by modern computers, would then be "much better" than the achievement of the great Chinese scholar. Any striving for useless or irrevelant precision becomes irrational.

Stevin presented both a geometrical and an arithmetical procedure for his monochord (Fokker 1966:442-49), and he did not shrink from multi-digit calculations; in fact, he demonstrated his figuring method with examples of 23- and 46-digit figures. But in tabulating his final monochord for practical application he stopped at four digits and used such shortcuts as proportional calculations (rule-of-three) for some of the high-powered roots. Even so Stevin's maximum error stays below 0.5 cents per semitone, i.e. undetectable by the finest musical ears. In other words, he knew the limits of practical usefulness and did not waste time or effort on achieving greater precision than necessary for rational purposes. Clearly, he showed more insight into the psycho-acoustical and tuning circumstances of the problem than Chu with his nine-digit tabulations.

CHINA

The earliest Chinese source containing a modification or manipulation of the traditional "Pythagorean" figures is the *Huai Nan Tzu,* The Book of Liu An, Prince of Huai Nan, ca. 122 B.C. (see Appendix No. I and Table I, col. 1).[10] The deviations from "Pythagorean" norm are caused by the fact that *Huai*'s values are limited to two-digit numbers and that his starting tone *Huang Chung* equals 81 units. As a consequence the change of only one unit in a pitch number creates differences between 23 and 41 cents, depending on the interval size, and on whether or not the last digit has been raised one unit in rounding off results as compensation for a cut third digit. One gets the impression that this monochord was conceived as a practical guide for the manufacture, tuning and playing of stringed instruments where string length measurements of more than two digits made little sense. Since all of *Huai*'s deviations shift the pitches from the sharp towards the flat side of intonation (e.g. from D♯ towards E♭), the net effect is one of consistent direction of pitch manipulation; if one is inclined to use a generous interpretation of the term, this monochord may be called an irregular tempering procedure.

Next in line comes Ching Fang, a diviner, mathematician, astronomer and acoustician of the Former Han Dynasty (202 B.C.-9 A.D.) who flourished around 45 B.C. He extended the traditional up-and-down principle from 12 to 60 steps of perfect fifths and fourths, creating a spiral of five arcs defining 60 microtonic intervals. Selecting the 12 pitches among the 60 which came closest to the quantities of an equal temperament then only dimly surmised, he achieved a creditable approximation to the theoretically correct values (see Table I, col. 2).

It seems that, so far, no Western or Chinese author has given Ching Fang the deserved credit for this achievement. Especially in China he has been severely criticized for supposedly attempting the senseless and impossible: to eliminate the ditonic comma by continuing the spiral of fifths to 60 steps. Did this ignorant man not realize that no power of 2 could ever be equal to any power of 3, his formula being, for the 60th step: $(\frac{2}{3})^{60} \times 2^{35}$? But our scholar had a different objective in mind: to reduce the comma and create, by approximation, an almost closed system of 12 not too unequal semitones (alternatingly 110 and 94 cents). In fact, at step no. 54 out of 60 he found that the comma had been reduced from 24 to 3.59 cents. He must therefore be credited with the invention of the 53-division of the octave which gained theoretical importance in Europe as late as the 17th century (Marin Mersenne, Athanasius Kircher.)[11] Moreover, Ching Fang was modern enough as a scientist to test his theoretical results by experiment. His measuring instrument *chun*, especially designed for the purpose, was actually more efficient than the *chun* used after his example 1600 years later by Prince Chu.[12] Thus, any superiority feelings of Ching Fang's critics were out of place.

Ch'ien Lo-chih, Astronomer-Royal and mathematician of the Liu Sung Dynasty (fl. 415-455 A.D.) is another important contributor. He continued Ching Fang's spiral from 60 to 360 steps, thereby reducing the comma to 1.845 cents and gaining a series of 12 semitones selected from the 360 pitches which form a truly excellent approximation to the theoretical values of equal temperament. (See Table I, col. 4). Most authors discussing his work give him the same treatment as Ching Fang usually received, and for the same reason: how stupid must Ch'ien have been, if he had to figure 360 steps of perfect fifths and fourths before realizing that the powers of 2 and 3 never meet. Anyway, his five maximum deviations from "ideal" equally tempered pitches are between 1.77 and 1.07 cents, while all other errors are of the order of 0.5 and 0.75 cents. This is so close to maximum tuning accuracy by the finest craftsmen that, from a practical point of view, no improvement is possible or necessary. Ch'ien Lo-chih's formula for step no. 360 reads $\frac{2}{3}^{360} \times 2^{209}$, and the agony of his figuring task becomes evident if we spell out the numbers Ching Fang had to handle at step no. 54 (Courant 1913:88):

$$176.177 \times \frac{50.437.239.049.155.671.823}{109.418.989.131.512.359.209}.$$

We do not care to visualize the figures at step no. 60; and for all we know, Ch'ien's numbers at step no. 360 may have been two city blocks long, unless he devised some efficient method of cutting hundreds of decimals without sacrificing reasonable accuracy.

Before Ch'ien another astronomer had already shown that there are no possible coincidences between powers of 2 and 3. He was Ho Ch'êng-t'ien (370-447 A.D.), and since he anteceded Ch'ien by almost a century, we can be assured that the later scholar was fully aware of this numerical impossibility. Ho contributed himself another approximation with quite acceptable results. Using the method of linear corrections, his monochord shows its largest deviations from theoretical pitches at 15.9 and 7 cents, while his smallest errors amount to 4.3 and 2 cents. (See Table I, col. 3).[13]

There were at least two more scholars who made valuable contributions to the field of equal temperament: Ch'en Chung-ju, an acoustician who flourished around 516 A.D., and Wang P'o, a famous Taoist scientist and engineer (fl. 959). It is not absolutely certain, but highly probable, that Prince Chu knew the work of all these predecessors when he began his studies. Furthermore, he may have known other works which have been lost since the turn of the 16th century. Considering all these earlier developments, it is impossible to credit Chu with the "invention" or "discovery" of equal temperament. The problem as such had been known in Chinese musical theory for 1700 years before Chu, and a substantial number of fair or good approximations, by a variety of methods, had been offered up to the 10th century A.D. Then, apparently, interest in the problem died down because developments in the nation's musical practice did not require good temperament solutions. When the Prince took up his studies, he was apparently working in a vacuum of 600 years which made it possible for him to claim a great innovation. Just as Stevin in Europe, Prince Chu was the *Errechner* (calculator) of this temperament, not the inventor.

EUROPE

In Western antiquity, Aristoxenos (4th century B.C.) may have given the first precise definition of equal temperament: he offered values representing certain seemingly equal parts or divisions of the octave, but it is unclear what these "parts" actually are. The case is quite controversial, but one particular interpretation leads to the conclusion that equal temperament is actually defined in these numbers. Barbour, among others, refuses to accept this interpretation and points out that several other interpretations are just as possible (Barbour 1951:1-2, 22-24).

The construction of one or more mean proportionals between two given values cannot be solved by Euclidian geometry, but according to Vitruvius two non-Euclidian solutions had been found by Archytas (c. 380 B.C.) and Eratosthenes (c. 230 B.C.):

> Now let us turn our thoughts to the researches of Archytas of Tarentum and Eratosthenes of Cyrene. They made many discoveries from mathematics.... For example, each in a different way solved the problem enjoined upon Delos by Apollo in an oracle, the doubling of the number of cubic feet in his altars; this done, he said, the inhabitants of the island would be delivered from an offence against religion. Archytas solved it by his figure of the semicylinders; Eratosthenes, by means of the instrument called the mesolabe (Vitruvius 1960:255).

The use of the *mesolabium,* a mechanical tool, for the construction of mean proportionals was well known during the Renaissance, and probably much earlier. A controversy in the 16th century as to whether the mesolabium could be used for constructing more than one mean proportional was settled by Stevin himself who, in his *Spiegeling der Singconst,* demonstrated how various mean proportionals can be achieved with the mesolabium technique or with a related geometrical method[14] (Fokker 1966:442-43).

Other Renaissance studies greatly advanced the knowledge of tempering. In 1523 Pietro Aron's *Toscanello in Musica* had appeared, with a detailed description of his ¼-comma meantone temperament. Before him Franchinus Gafurius, in *Practica Musica* (1496) reported how organists tuned their fifths by a small, indefinite amount too low, naming that procedure "participata," i.e., temperament (Barbour 1951:25). Zarlino, in 1571, called meantone tuning "a new temperament," thereby demonstrating that other types of temperament had existed earlier. And since Gafurius described the organists' tuning practice not as a new fashion, but as a current custom, the conclusion is that a temperament based on slightly lowered fifths must have existed in Europe decades earlier than 1496. This may easily have been equal temperament (with its fifths lowered by 2 cents), an assumption well supported by a number of paintings of Ercole de Roberti (1450-1496) in the National Gallery in London. In these pictures the frets of lutes and viols are apparently placed at equal temperament distances; a number of other, especially Flemish, paintings created between 1492 and 1533 also show fretted string instruments with the same temperament distances (Barbour 1951:12). From this and other evidence, including high-ranking irregular systems that approach equal temperament, Barbour concludes:

> It is easy to believe, therefore, that organs were tuned as well in 1500 as they are generally today (1951:25).

Finally, the best approximation in European temperament theory was proposed in 1581 by Vincenzo Galilei who introduced the ratio 18:17 for the tempered semitone; this equals 99 cents, 1 percent short of the theoretically correct value and still below the threshold of pitch discrimination. This ratio antedated, by several years, the solutions of Stevin and Chu, and although it was only a numerical approximation, it certainly qualifies as an appropriate quantitative definition of equal temperament. Since we may, thus, be sure that in Europe this temperament, or close approximations to it, were widely used a hundred years before Chu and Stevin, we must conclude without prejudice that neither of them can be credited with a priority of invention or discovery.

The reasons for Robinson's misjudgement of the priority question become evident in the following somewhat perplexing statements:

> Certainly a time was coming [at the end of the 16th century] when Europe could profit from such an invention.

and:

> ... it is striking that *so little can be ascertained* about [equal temperament's] European origin when *everything is known* about its invention in China (Italics added) (Robinson 1962:224-25).

Lack of information about European temperament developments, regrettably, is as common as it is distressing among authors, both east and west. Barbour's historical survey of 1951 (11 years before publication of the Robinson essay) contains more than 150 different tunings and temperaments representing dozens of approaches and systems. Among these are nearly 40 solutions and propositions for equal temperament besides those of Prince Chu and Stevin, with quite a few of them antedating these two scholars. The volume also offers a bibliography of close to 220 titles dealing with the topic, a majority of them concerning the European history of equal temperament. But even before Barbour's convenient and comprehensive work most of these titles were easily accessible.

Without this background information Robinson's view of Western developments had to remain inadequate and misleading. In particular, he could not have seen that both Stevin's and Chu's achievements were completely useless

TABLE I. HIGHLIGHTS OF TEMPERING DEVELOPMENTS IN CHINA

Col. 1 Original Huai Nan Tzu c. 122 B.C. Two-Digit Monochord					Col. 2 Ching Fang c. 45 B.C. 60- or 53-Division		Col. 3 Ho Ch'êng-T'ien 370-447 A.D. Linear Correction		Col. 4 Ch'ien Lo-Chih c. 450 A.D. 360-Division	
Pitch Name: Chinese	Pitch Name: Western	Pitch Value	Cents: Huai	Cents: Pythagor.	Step No.	Cents	Tone	Cents	Step No.	Cents
Huang Chung	C	81	0	0	0	0	C	0	0	0
Lin Chung	G	54	702	702	48	93.84	C♯	101	154	101.07
T'ai Ts'u	D	72	204	204	2	203.91	D	200	255	198.53
Nan Lü	A	48	906	906	50	297.75	D♯	297	103	301.36
Ku Hsi	E	64	408	408	4	407.82	E	398	204	398.82
Ying Chung	B	43*	1096*	1110	52	501.66	E♯	493	358	499.89
Jui Pin	F♯	57	608	612	47	591.88	F♯	596	153	599.12
Ta Lü	C♯	76	110	114	1	701.96	G	699	307	700.19
I Tsê	G♯	51	801	816	49	795.80	G♯	791	102	799.41
Chia Chung	D♯	68	303	318	3	905.86	A	897	256	900.48
Wu I	A♯	45	1018	1020	51	999.70	A♯	985	51	999.70
Chung Lü	E♯	60	520	522	5	1109.78	B	1091	205	1100.78
Huang Chung	c	—	———	1224	53	1203.62	C′	1200	306	1198.23
Max. Dev.			20.0	24.0	Max. Dev.	9.78	Max. Dev.	15.0	Max. Dev.	1.47
Tot. Dev.			84.0	156.0	Tot. Dev.	52.02	Tot. Dev.	54.0	Tot. Dev.	8.41
Mean Dev.			7.0	13.0	Mean Dev.	4.73	Mean Dev.	4.91	Mean Dev.	.76

*The original Huai Nan Tzu gives Ying Chung (B) wrongly as 42, with the cents equivalent of 1137. Chu Tsai-yü corrected this obvious error in the original into 43 = 1096 cents. Amiot also quotes 43 from Chu's source.

All deviations above refer to equal temperament values.

Col. 5

Chu Tsai-Yü, 1584, as intentionally misattributed to Huai Nan Tzu

750:749 Tempering = 2.31 cents

Pitch Name: Chinese	Western	Cents
Huang Chung	C	0
Lin Chung	G	699.69
T'ai Ts'u	D	199.38
Nan Lü	A	899.07
Ku Hsi	E	398.76
Ying Chung	B	1098.45
Jui Pin	F♯/G♭	598.14
Ta Lü	C♯/D♭	97.83
I Tsê	A♭/G♯	797.52
Chia Chung	E♭/D♯	297.21
Wu I	B♭/A♯	996.90
Chung Lü	F/E♯	496.59
Huang Chung	c	--.--
Max. Dev.		3.72
Tot. Dev.		20.66
Mean Dev.		1.88

Col. 6a

Chu Tsai-Yü, 1584
749.153.538 Tempering

Pitches in original circular order

Pitch Name: Chinese	Western	String Lengths
Huang Chung	C	10.000 inches
Lin Chung	G	6.674.199.27
T'ai Ts'u	D	8.908.987.18
Nan Lü	A	5.946.035.57
Ku Hsi	E	7.937.005.25
Ying Chung	B	5.297.315.47
Jui Pin	F♯/G♭	7.071.067.81
Ta Lü	C♯/D♭	9.438.743.12
I Tsê	A♭/G♯	6.299.605.24
Chia Chung	E♭/D♯	8.408.964.15
Wu I	B♭/A♯	5.612.310.24
Chung Lü	F/E♯	7.491.535.38
Huang Chung	c	5.000 inches
Max. Dev.		0 cents
Tot. Dev.		0 cents
Mean Dev.		0 cents

Col. 6b

Chu Tsai-Yü, 1584
Same monochord as in Col. 6a
with pitches re-arranged
in chromatic order

Pitch Name: Chinese	Western	String Lengths
Huang Chung	C	10.000 inches
Ta Lü	C♯/D♭	9.438.743.12
T'ai Ts'u	D	8.908.987.18
Chia Chung	E♭/D♯	8.408.964.15
Ku Hsi	E	7.937.005.25
Chung Lü	F/E♯	7.491.535.38
Jui Pin	F♯/G♭	7.071.067.81
Lin Chung	G	6.674.199.27
I Tsê	A♭/G♯	6.299.605.24
Nan Lü	A	5.946.035.57
Wu I	B♭/A♯	5.612.310.24
Ying Chung	B	5.297.315.47
Huang Chung	c	5.000 inches
Max. Dev.		0 cents
Tot. Dev.		0 cents
Mean Dev.		0 cents

All deviations above refer to equal temperament values.

TABLE I. HIGHLIGHTS OF TEMPERING DEVELOPMENTS IN CHINA to musical practice in Europe as well as in China: in the West the system had been known and used long before Stevin, and Chinese music had developed in directions where temperaments were superfluous. Furthermore, in the West musicians were uninterested in monochords or a twelfth root of 2; they tuned their instruments by observing and counting beats. It would have to be an unusually scholarly and historically interested keyboard tuner who ever saw a "monochord," both the measuring device or the tabulations of 12 string lengths and similar pitch definitions. Here we have, in all probability, the reason why Stevin never undertook to publish his essay.

SELECTED WORKS

(18 of 21 preserved titles)

In order to give a better description of the contents of the following 18 titles, we have translated them quite freely and, in various cases, in deviation from previous bibliographical custom. In particular the phrase *Lü Lü*, represented by two different words and characters, has been translated throughout as "The Semitones" or "The Twelve Semitones," because that was the actual meaning of the phrase at the time of Chu Tsai-Yü's writing.

A. About Mathematics and Calendrical Science:

1. *Lü Li Yung T'ung* (= The Twelve Semitones and their Calendrical Coordination). One of the two titles contained in no. 2. 1581.
2. *Li Shu* (= Work of the Calendar). c. 1595, printed 1601.
3. *Suan Hsüeh Hsin Shuo* (= New Report about Measuring Methods), (i.e. in Acoustics and Music). 1603.

B. About Musical Theory and Temperament:

4. *Lü Hsüeh Hsin Shuo* (= New Report about the Theory of the Twelve Semitones). 1584.
5. *Lü Shu* (= Work about the Twelve Semitones), in two parts of which the first is the following famous title:
6. *Lü Lü Ching I* (= Refined Interpretation of the Twelve Semitones). 1595 or 1596. Probably not published before 1606.
7. *Lü Lü Cheng Lun* (= Formal Discussion of the Twelve Semitones). Before 1606.
8. *Lü Lü Chih I Pien Huo* (= Defense of his Writings on Musical Theory). 1610.

C. About Musical History:

9. *Yüeh* (or: *Yo*) *Hsüeh Hsin Shuo* (= New Report about Musical History). Before 1606.

10. *Ts'ao Man Ku Yüeh* (or: *Yo*) *P'u* (= Treatise about Melodies of Antiquity, with Accompaniment). (Including music for *Ch'in,* the seven-stringed half-tube zither, in ancient notations). Before 1606.

D. About Dance and Dance Music:

11. *Hsüan Kung Ho Yüeh* (or: *Yo*) *P'u* (= Treatise about Melodies and their Transposition). (Illustrated by the formal dance music for the Ode *Kuan-chui*). Before 1606.
12. *Liu Tai Hsiao Wu P'u* (= Treatise about the Dances of the Six Dynasties of Antiquity). Before 1606.
13. *Ling Hsing Hsiao Wu P'u* (= Treatise about the Rural Dances for the Ceremonies in the Temple of Agriculture). Before 1606.
14. *Hsiao Wu Hsiang Yüeh* (or: *Yo*) *P'u* (= Treatise, with Instructions, Music and Instruments for the Dances in nos. 12 and 13). Before 1606.
15. *Erh I Chui Chao T'u* (= Coordinated Choreography for Dance Groups of Two Pairs each). Before 1606.

E. Miscellanea:

16. *Sê P'u* (= Treatise about the *Sê*, the Half-tube Zither of Antiquity). Before 1606.
17. *Hsiang Yin Shih Yüeh* (or: *Yo*) *P'u* (= Treatise on the Music for the Odes performed at District Banquets). Before 1606.
    Note: The titles dated "Before 1606" were written earlier. They were submitted as a collection to the emperor in 1606 after the engraving, especially of the many illustrations, had caused substantial delays in publication.

F. Collective Work:

18. *Yüeh* (or: *Yo*) *Lü Ch'üan Shu* (= Collected Writings about Music and the Twelve Semitones). Posthumous, c. 1620. It contains the above titles no. 4, 6 and 3.

APPENDIX I

Prince Chu's Calculation Methods

The traditional method of Chinese antiquity to calculate the 12 semitones of their ("Pythagorean") tone system is the up-and-down principle: one perfect fifth up, one perfect fourth down alternatingly. The ratios of these intervals being 2:3 and 4:3, respectively, the traditional untempered system is thus defined by the series $1 \times 2/3 \times 4/3 \times 2/3 \times$

4/3 . . . etc.[15] This sequence, it will be recalled, ends up, after 12 steps, with the tone B♯, represented by the ratio

$$262.144 : 531.441 = 0.49327.$$

This pitch is slightly higher than the upper octave 0.5 of the value 1.0 with which the whole sequence started. The resulting difference by which the circle of fifths fails to close completely, is represented by the ratio

$$524.288 : 531.441 = 0.98654,$$

a microinterval called the "Pythagorean" or ditonic comma. In logarithmic terms the comma is usually given as 24 cents (more precisely 23.46 cents). The essence of all equal-temperament solutions is thus, briefly, the elimination of the comma while creating, at the same time, 12 semitones of equal size. This procedure is very simple when using modern logarithmic methods and measuring units, such as Ellis cents: 1,200 cents for the perfect octave, 702 cents for the perfect fifth, 498 cents for the perfect fourth, 100 cents for each equally tempered semitone, 700 and 500 cents, respectively, for the equally tempered fifth and fourth. All that has to be done in this kind of calculation procedure is to reduce (i.e. temper) each fifth by the microinterval of 2 cents, from 702 to 700 cents, and by increasing the size of the fourths from 498 to 500 cents.

Before the invention of logarithms[16] and interval calculation in cents (by Alexander John Ellis in 1884-85) such procedures were very difficult and extremely time-consuming, producing many crude or, occasionally, refined approximations to the as yet unknown precise values of equal temperament or other tempering systems, and using a variety of mechanical, geometric or arithmetic methods.

In both his 1584 and 1595/96 works (nos. 4 and 6) Chu ascribes the invention of a most ingenious and efficient approximation to equal temperament especially to *Huai Nan Tzu,* The Book of Liu An, Prince of Huai Nan (c. 122 B.C.),[10] and to the *Chin Shu*[17] and *Sung Shu.*[18] Realizing that in the old up-and-down principle the comma was caused by slightly too large fifths and slightly too small fourths, Chu adjusted both intervals by small amounts through all 12 steps of the operation. Instead of using the original ratios 2:3 ($= \frac{500}{750}$) and 4:3 ($= \frac{1000}{750}$), he introduced the fractions

$$\frac{500}{749} \text{ and } \frac{1000}{749},$$

thus tempering the fifths and fourths by a small quantity. This adjustment equals 2.31 cents, amazingly close to the 2 cents of theoretically precise equal temperament: the fifths are 699.69 cents wide instead of 700 cents, the fourths 500.31 cents, instead of 500 cents! (For the complete monochord see

Table I, col. 5). It should be noted here, that the cents tabulation of this monochord as given by Reinhard is wrong throughout, owing to a misinterpretation of the objective and of the procedure ascribed by Chu to *Huai Nan Tzu*.[19]

Before proceeding, we must state with all possible emphasis that Huai was not the originator of this 749-temperament. There is no mention or even implication of the 749-method in the *Huai Nan Tzu*. Furthermore, the limitation of Huai's numbers to two digits (see Table I, col. 1) makes a distinction between the divisors 750 and 749 impossible; at least four digits are required to show a difference between applications of the two divisors. This means, of course, that we are dealing here with an intentional mystification of the reader by Prince Chu. Specifically he says, in discussing the various traditional calculation methods of antiquity:

> The third [of four] methods comes from the *Huai Nan Tzu, Chin Shu* and *Sung Shu*.
>
> ... The method of *Huai Nan Tzu* is not the 2/3, 4/3 method ... therefore its *Lü* numbers are quite different.
>
> ... The second method comes from the book of *Huai Nan Tzu*; the *Chin Shu, Sung Shu* and the references of Ts'ai Yüan-ting [1135-1198].[20]

Besides there being no trace of the 749 divisor in the *Huai Nan Tzu*, there is no mention of it in the *Sung Shu* or in the work of Ts'ai Yüan-ting. A careful search of the other dynastic histories up to Prince Chu's Ming Dynasty, however, reveals one single occurrence of the divisor 749 in the history of the Chin Dynasty, completed in 635 by Fang Hsüan-ling. This work contains in one of its treatises a highly unorthodox monochord for a set of very long *Ti* flutes.[21] Among its various departures from traditional procedures (such as calculations to the flat side of the circle, e.g., e♭ and a♭ instead of d♯ and g♯), there is the very first step from *Huang Chung* to *Lin Chung* (C to G) figured with the divisor 749, and as a fourth downward instead of the usual fifth up. The system, attributed to the acoustician Hsün Hsü (+289), does not mention, in the *Chin Shu* report, the divisor 749; Prince Chu must have discovered it the same way the present writer did: by calculating how the various numbers of this unusual monochord were arrived at. It is significant that the divisor 749 is used only once in the Hsün Hsü monochord, on the very first step C-G; all further steps are either calculated in the traditional 2/3, 4/3 pattern, or they show other deviations unrelated to the 749 division.

Prince Chu certainly knew of this monochord because any Ming scholar involved in work on the nation's tone system had to study the various dynastic histories for reports on the topic. Thus, he must have discovered very soon what the divisor 749, when consistently applied to a complete monochord, would do to a circle of fifths and fourths. Why he decided to credit the origins of his own tempering method to the *Huai Nan Tzu* with which it had nothing to do, is a matter of speculation. We believe that Chu wanted to cover himself against a possible charge by ardent traditionalists that he advocated a tuning system at odds with a tradition of 17 centuries. Attribution to such time-honored and highly respected sources as *Huai Nan Tzu* (and, possibly, the *Chin Shu*) provided a safety valve he may have felt he needed, as he would experience a few years later when his treatise on calendrical reforms was rejected on traditionalist grounds: the official government specialists on musical acoustics and theory were just as conservative as the imperial astronomers and calendar experts.

Actually, the superimposition of the 749-method on the *Huai Nan Tzu* figures was an amazingly clever scheme of Prince Chu to render his attribution fully convincing; the limitation of Huai's monochord to two digits made a discovery of the deception practically impossible. Furthermore, the Prince designed a highly sophisticated figuring procedure involving quasi-mystical tricks of number theory which must have endeared the sequence below to all *Huai Nan Tzu* admirers; for Huai was the most famous of all Han Dynasty Taoists and enjoyed the reputation of a great mystic, magician and adept of all obscure arts. Any surprising scheme of numerical mysticism would be precisely in keeping with what one would expect of Huai:

Huang Chung (C) = 81.
    81 X 500 = 40.500 : 749 = 54, remainder 54;
Lin Chung (G) = 54.
    54 X 1000 = 54.000 : 749 = 72, remainder 72;
T'ai Ts'u (D) = 72.
    72 X 500 = 36.000 : 749 = 48, remainder 48;
Nan Lü (A) = 48.
    48 X 1000 = 48.000 : 749 = 64, remainder 64;
Ku Hsi (E) = 64, . . . etc. through the remaining seven steps of the circle of fifths.

This continuous coincidence of quotient and remainder, caused by the interaction of the initial figure 81 and the divisor 749, is a most cunning bit of number theory which testifies to Prince Chu's skills. It is, in fact, so sophisticated that it cannot really be attributed to the time of *Huai Nan Tzu* (c. 122 B.C.) when the science of arithmetics in China was just not developed to that level. For readers interested in the particular properties of the number 749, Kaufmann has published a specialized study which, unfortunately, still accepts Huai's authorship of this arithmetical *tour de force* and does not seem to be aware of Chu's role in the construction of the 749-method (Kaufmann 1969:371-82).

It is hardly necessary to repeat that no mention or trace of this whole procedure occurs in the *Huai Nan Tzu*. Obviously, Chu picked up the

749-divisor in its solitary occurrence in the *Chin Shu,* investigated its numerical potential and found that it reduced the size of a perfect fifth by very nearly the correct amount needed for an almost closed scale system of tempered fifths and fourths. While playing around with the numbers 81 and 749, he apparently discovered the quotient-remainder coincidences and decided upon his historical mystification: to implant the whole scheme on Huai's monochord where it could not be detected. Table I, col. 5, shows the excellent approximation of a monochord calculated with the unrefined divisor 749, as wrongly attributed by Chu to Huai. When Chu found out how close he had come even with the integer number 749, his ambition was triggered and he set out for additional decimals. Table I, col. 6 tabulates his final nine-digit monochord based on the refined divisor 749.153.538 as published in 1584 (in treatise no. 4).

If we line up these numbers in the customary Chinese way—i.e., in the circular order of the up-and-down method—the last item becomes *Chung Lü* = E♯, and the higher octave, step no. 12, is left blank. This arrangement makes it immediately evident that the figures for the final circular step must be

$$749.153.538 \times \frac{500.000.000}{749.153.538} = 500.000.000,$$

500 being the value of the upper octave in this monochord based on the octave ratio 10:5. It also becomes obvious that 749.153.538 is the divisor which applies to all 12 semitones (Table I, col. 6a).

If one looks at the same figures in the customary Western line-up of chromatic steps, these essential relations are obscured, and this apparently is the reason why Robinson failed to discover the actual procedure behind this nine-digit monochord of 1584. Instead, he speculates erroneously (Robinson 1962:223-24) that Chu figured *Lüs* no. 7,4 and 10 as single mean proportionals between *Lüs* 1 and 13 which he assumes to be one foot and ½ foot in length (although Chu's actual dimensions in the 1584 monochord were 10 and 5 [units], respectively.) The remaining pitches, our author believes, were to be figured by applying various powers of the twelfth root of 2. The trouble is clearly that the Western mind is too prejudiced to imagine another method of equal temperament calculation.

Robinson is partly correct, however, in quoting Chu from chapter 1 of his *Lü Hsüeh Hsin Shuo* (1584) to the effect that each string [*Lü*!] in turn "must be divided by the figure for double-length *Ying-Chung* (i.e., B, lower octave) . . . which is a way of getting the pitches in their serial order." This quotation is introduced by Robinson with the statement that the figure 1.05946, being the value of $\sqrt[12]{2}$, was "of course (sic!, emphasis added) obtained by Chu Tsai-yü for the note immediately below his standard *Huang-Chung* length."

Again, the Western mind is preprogrammed to look out for the "magic" number 1.05946 as the only irrefutable proof that a twelfth root of 2 has been extracted and used for a calculation procedure. In fact, however, Prince Chu obtained this figure not by extracting that root, but again by using the divisor 749, apparently with no more than two additional decimals. The 1584 work contains a 36-item monochord through three octaves in ascending pitch order, with the ratios 200:100:50:25, which reads in part:

No.	Pitch Name	Western Pitch	Length of Lü
1	Huang Chung	C	200.00
2	Ta Lü	C♯/D♭	188.77
3	T'ai Ts'u	D	178.17
4	Chia Chung	D♯/E♭	168.17
5	Ku Hsi	E	158.74
6	Chung Lü	F	149.83
7	Jui Pin	F♯/G♭	141.42
8	Lin Chung	G	133.48
9	I Tsê	G♯/A♭	125.99
10	Nan Lü	A	118.92
11	Wu I	A♯/B♭	112.24
12	Ying Chung	B	105.94
1	Huang Chung	c	100.00
2	Ta Lü	c♯/d♭	94.38
etc.	etc.	etc.	etc.

The above sequence was unquestionably compiled by the alternating use of the factors

$$\frac{500.00}{749.15} \text{ and } \frac{1000.00}{749.15}$$

which led, in five consecutive operations, to the value 105.94:

C	G	D	A	E	B
200.00	133.48	178.17	118.92	158.74	105.94.

A quick verification reveals that the use of the divisor 105.94 for construction of the above chromatic monochord produces slightly different results, with deviations of up to two units in the second decimals. This proves that the divisor used by Chu for this monochord was 749.15.

At this point several conclusions can be drawn:

1–The crucial figure 1.0594 . . . does not depend on the extraction of the twelfth root of 2: it has been established by the much simpler procedure of five consecutive applications of the 749-method.

2–The semitone step from *Ying Chung* (B) = 105.94 to *Huang Chung* (c) = 100.00 gave Chu the value of an equally tempered semitone without recourse to any twelfth root.

3–The number 2 does not play any exclusive or even important role in Chu's calculations since, in his earlier work, none of his monochords are based on the octave ratio 2:1. The treatise of 1584 uses the ratios 10:5 and 200:100:50:25. Therefore the crucial figure showing up in the above chromatic construction is 105.94 instead of 1.0594. . . ; in the circular monochord (Table I, col. 6) the *Ying Chung* number (B) is 5.297.31 instead of 1.059.462, i.e., five times the crucial figure, because the octave ratio applied here is 10:5, i.e., five times the ratio 2:1. This strongly suggests that in 1584 Chu never used the twelfth root of 2 nor was he aware that he was using that root when dealing with the figure 105.94.

As we shall see in Appendix no. II, the later work of 1595/96 uses a variety of divisors, all of them based on, and produced by, the 749-method, and all of them–except one–constituents of the original nine-digit monochord of 1584. It is highly instructive to have a closer look at these various divisors.

The first of them is *Chung Lü* (F) = 749.153.538. This is, of course, the original construction value which receives its place in the monochord line-up as the pitch number for the tone F. In connection with the factor 500.000. . ., it produces a circular up-and-down scale with the sequence C–g–d–a–e . . . etc., and the octave ratio 10:5.

Next comes *Ying Chung* (B) = 529.731.547. Its application builds, in connection with the factor 500.000 . . . , a chromatic ascending scale with the sequence C–C♯–D–E♭–E . . . etc., octave ratio 200:100 or 2:1, depending on the desired position of the decimal point. (Note: Divide all 12 semitones of the 10:5 monochord by 5 and get the 12 semitones of the 2:1 monochord).

The third divisor is *Ta Lü* (C♯ or D♭) = 943.874.312. Its application creates a descending chromatic scale, octave ratio 200:100, with the sequence c–B–B♭–A–A♭. . . , etc. (Note: The first step, 10.000 . . . inches divided by 9.438.743.12, gives *Ying Chung* (B) = 1.059.463 = $\sqrt[12]{2}$. Here is yet another way to establish this crucial value without extracting any twelfth root.

Divisor no. 4 is *Lin Chung* (G) = 667.419.927. Its application, in combination with the factor 500.000 . . . , yields a circular monochord with the octave ratio 10:5 (or 1.000:500, respectively), to the flat side of the circle, in the sequence C–f–b♭–e♭–a♭ . . . , etc.

Finally, there are in the *Lü Lü Ching I* (1595/96) two instances of the divisor 1.059.463.094 = $\sqrt[12]{2}$, and two applications of the divisor 1.029.302.236 = $\sqrt[24]{2}$. Both values are, significantly, not used in the calculation of string lengths, but for figuring measurements of pitch pipes, such as pipe lengths, outer circumferences, inner diameters and other cylindrical parameters. And, significantly, these four calculations are based on pipe lengths of two feet and one foot, i.e., the octave ratio 2:1 which may involve the twelfth or twenty-fourth root of 2.

Yet, even these specific examples do not offer any proof that Prince Chu operated here knowingly or intentionally with the twelfth root of 2, or that he ever extracted that root in order to use it for his calculations. The fact is clearly that all the various divisors applied by Chu could be, and were, established by the much simpler and equally precise 749-method which made the tedious root extraction unnecessary. On the other hand, there is no conclusive evidence that Chu never extracted a nine-digit twelfth root of 2. Judging the facts without prejudice either way, one would have to state that Chu may possibly have established and knowingly used that root, but it is very unlikely. The many formulae and definitions given in Appendix II seem to be closely related to the expression $\sqrt[12]{2}$; but they display that relationship only because the construction of each 12-semitone monochord involves twelve consecutive divisions by the same number, a procedure which is, naturally, defined by some twelfth root.

The question as to why Chu never gave an explicit explanation of his calculation methods, cannot be answered by the assumption that he wanted to conceal or, at least, obscure them. It will be recalled, e.g., that he actually volunteered some pertinent information: "divide by the number for *Ying Chung* (B) in order to create a chromatic monochord." The answer is rather to be found in some fundamental differences between Western and Chinese scientific approaches and objectives. European science started quite early to develop theoretical bases for procedures, while Chinese scholars were mainly interested in practical solutions gained by empirical approaches. In this way Chinese sciences excelled in many important inventions and achievements which the theoretically inclined Occident did not produce until much later. This is true for the field of mathematics as well, and Needham has very convincingly described the differences between Chinese and Western mathematical methods:

> . . . Chinese mathematicians never spontaneously invented any symbolic way of writing formulae, and until the time of the coming of the Jesuits, mathematical statements were mainly written out in characters. Strangely, in a people who carried algebra so far, the equational form *remained implicit*, and there was no indigenous development of an equality sign (=). How far the widespread use of the counting-board and abacus acted as an inhibiting factor is a moot point; they certainly allowed calculations to *vanish without trace*, leaving no record of the intermediate stages by which the answer was reached (Italics added) (Needham 1959:152).

The Western scientific mind is justifiably proud of Stevin's theoretical definition of equal temperament: eleven mean proportionals between the numbers 1 and 2, and the twelfth root of 2 for the size of the tempered semitone. But it comes as a surprise or even a shock to us that the Chinese mind achieved the same results and a good deal more without the benefit of theoretical definitions and without the tedium of extracting a twelfth root, by the ingenious device of consecutive divisions. Moreover, Chu's solutions offer a variety of traditional octave ratios beyond the limited and exclusive Western ratio 2:1. From a practical point of view the strictly empirical Chinese solution must be termed superior, both for its efficiency and its simplicity. Furthermore, the introduction of the many different divisors by Chu revealed a variety of hidden relationships between the individual tempered semitone pitches—correlations that were undiscovered in Western musical theory up to this day.

There are two questions on Prince Chu's calculation methods which remain unanswered, thus inviting speculation: the first concerns the six additional decimals he added for greater precision to the original divisor 749; the second is whether and how Chu could have extracted a twelfth root of 2 if he had really wanted to do so.

For the determination of the six decimals a simple application of "brute force" with successive guesses would produce a serviceable solution, with each guess being the average of the two previous guesses. In all, 36 steps at 12 operations each, = 432 operations, are needed to establish six correct decimals. For a Chinese mathematician skilled in the daily use of the swift-working abacus this procedure does not seem to be cause for real terror, and it may well be that this was the method used by Prince Chu. If we consider that the aristocratic and financial standing of the Prince would have permitted him to employ educated clerks or assistants who may have performed such mechanical tasks for him, the use of this primitive method becomes even more probable.

A more sophisticated procedure would be "linear interpolation." It also entails approximation by successive guesses, each guess being the weighted average of the two previous guesses; one assumes that the function in question is linear and calculates an approximate value for the desired decimal, using the equation for a straight line. From Chu's published works it is evident that he was an accomplished arithmetician; although Chinese mathematical practice did not work with equations the way Western tradition does, it is conceivable that the Prince may have used some sort of linear interpolation or a comparable technique.

If we wish to assume that Chu actually wanted to extract the twelfth root of 2, he could have done it with techniques known in Chinese mathematical science of his time. In Western mathematics "Horner's Method" of 1819, also known as "synthetic division," offers an efficient procedure for the successive extraction of square and cubic roots. This method was anticipated in China by Ch'in Chiu-shao in 1247 in his work *Shu Shu Chiu*

Chang (Mathematical Treatise in Nine Sections). It has, in fact, its origins in a much earlier work of the Later Han Dynasty when the first skills in root extracting were developed.[22] "Horner's Method," thus, is actually a rediscovery and refinement of the earlier Chinese achievements (Wang and Needham 1955:345 ff). With this information and procedures from the work of Sung and Yuan Dynasty algebraists[23] Chu should have been in the position of extracting square, cubic and higher roots.[24]

While conceding this possibility for the sake of argument, we are nevertheless fully convinced that the Prince did not extract that twelfth root for use in his calculations because he had better and simpler procedures; in those cases where the number 1.059.463 turns up in the figuring, it had been obtained by ordinary division.

It is interesting to note that, as late as 1779, J. P. Kirnberger also hit upon an idea which is closely related to the 749-method, as an excellent approximation to equal temperament values. He used the ratio 10.935 : 8.192 for the tempered fourth which corresponds, in decimal terms, to 749.1541 . . .; this is better than the fake *Huai Nan Tzu* figure 749, but not as precise as Prince Chu's nine-digit refinement (Barbour 1951:64).

APPENDIX II

The Contents of the *Lü Lü Ching I* (1595/96) A Brief Annotated Survey

The *Lü Hsüeh Hsin Shuo* (= New Report about the Theory of the [Twelve] Semitones) of 1584 is important because of the equal temperament monochords it contains.[25] The *Lü Lü Ching I*, however, as its title indicates (Refined Interpretation of the [Twelve] Semitones), has a much greater importance for the history of musical theory. It represents a continuation and brilliant enlargement of the findings published in the earlier work, offering many new insights into the quantitative aspects of the temperament. Since so far none of Prince Chu's writings have been translated into any Western language, the following survey could serve, at least temporarily, to fill a regrettable gap in Western literature on the subject.[26]

As pointed out in Appendix I, Chinese mathematical practice did not develop the equational forms which dominate Western mathematics. For the reader's benefit we have constructed equations from the numerical and other quantitative information provided by Chu's text. It should be realized, however, that such unknown quantities as X or other variables do not occur in the Chinese original. We shall, for each example, state Chu's numerical definitions plus interpretive equations defining the algebraic procedures involved.

For brevity's sake we dispense with the repetition of the traditional Chinese *Lü* names (*Huang Chung, Ta Lü*, etc.) and give instead the Western pitch equivalents (C, C♯, etc.) for the twelve semitones. The fact, that the basic tone *Huang Chung* was close to the pitch of F during the Ming Dynasty, is conveniently disregarded.

LÜ LÜ CHING I

Chapter 1. Lengths of Strings and Pitch Pipes

Prince Chu presents first a refined version of the alleged *Huai Nan Tzu* method by correcting the divisor 749 into 749.153.538; obviously, he aims from the outset for a monochord precise to nine digits. Starting with a string length of 10 inches (or 10 units) for C (*Huang Chung*), he indicates the following expressions for the other semitones (in terms of numbers, not equations):

C = 10 (units)	G = $10\,X$	D = $20\,X^2$
A = $20\,X^3$	E = $40\,X^4$	B = $40\,X^5$
F♯ = $80\,X^6$	C♯ = $160\,X^7$	G♯ = $160\,X^8$
D♯ = $320\,X^9$	A♯ = $320\,X^{10}$	E♯ = $640\,X^{11}$
C = $1280\,X^{12}$.		

Chu defines: $X = \frac{500.000.000}{749.153.538}$.

(p. 14 b).

We have here a "circular" procedure using the up-and-down principle from *Lü* to *Lü* and multiplying alternatingly by X (500) and 2 X (1.000) in the successive powers of X. The principle is twice interrupted at steps 7 and 12 when doubling (i.e. octave transposition) of X is required. Algebraically the method is defined as follows:

Since $1280\,X^{12} = 10$, $X = \sqrt[12]{\frac{1}{128}} = \frac{500.000.000}{749.153.538}$.

(Kuttner-Kuo).[27]

The next section demonstrates a similar up-and-down procedure, starting with C = 10, ending with C = 10, i.e., a "circular" or "closed" system to the flat side of the circle. The 12 semitone expressions indicated are:

C = 10	F = $10\,X$	B♭ = $10\,X^2$	E♭ = $20\,X^3$
A♭ = $20\,X^4$	D♭ = $40\,X^5$	G♭ = $40\,X^6$	C♭ = $40\,X^7$
F♭ = $80\,X^8$	B♭♭ = $80\,X^9$	E♭♭ = $160\,X^{10}$	A♭♭ = $160\,X^{11}$
C = $320\,X^{12}$.			

Here the order of steps is reversed, beginning with the higher octave C and working down towards lower C.

Chu's definition: $X = \frac{500.000.000}{667.419.927}$.

Kuttner-Kuo:

Since $320\ X^{12} = 10$, $X = \sqrt[12]{\frac{1}{32}} = \frac{500.000.000}{667.419.927}$.

(p. 16 a.)

The third procedure abandons the up-and-down principle and "circular" method, by progressing from a lower octave C = 10 units, semitone after semitone, to the higher octave $C_1 \doteq 5$ units, octave ratio 10:5. Here, then, is for the first time a construction defining 11 mean proportionals between 10 and 5. It is not clear whether Prince Chu realized that he was defining mean proportionals, but his arithmetic procedure is correct:

$C = 10$ $\quad C\sharp = 10\ X$ $\quad D = 10\ X^2$ . . . etc.
$C_1 = 10\ X^{12}$

Chu defines: $X = \frac{500.000.000}{529.731.547}$.

Kuttner-Kuo:

Since $10\ X^{12} = 5$, $X = \sqrt[12]{\frac{1}{2}} = \frac{500.000.000}{529.731.547}$.

This is identical with $\frac{1.000.000.000}{1.059.463.094}$, and represents the reciprocal value of $\sqrt[12]{2} = 1{,}05946$ . . . .

(p. 18 b.)

The fourth proposition reverses the procedure of the previous one, starting out with the higher octave $C_1 \doteq 5$ (*Huang Chung, Pan Lü*)[28] and ending with the lower C (*Huang Chung, Chêng Lü*) = 10 units.

$C_1 = 5$ $\quad B = 5\ X$ $\quad A\sharp = 5\ X^2$ . . . etc.
$C = 5\ X^{12} = 10$.

Chu defines: $X = \frac{1.000.000.000}{943.874.312}$.

Kuttner-Kuo:

Since $5\ X^{12} = 10$, $X = \sqrt[12]{2} = \frac{1.000.000.000}{943.874.312}$.

(p. 20 a.)

So far, it can be assumed, Prince Chu had mainly string lengths in mind, and his various measuring units (10:5, or 2:1) were conceived mainly as monochord string lengths. He now proceeds to the traditional *Lü* (pitch-pipes) of antiquity where, besides pipe lengths, a variety of other dimensions have to be considered, thus making the accoustical conditions much more complex. The first calculation defines the "slanting length" of the 12 pipes; this probably means a diagonal cross-section, with the traditional ancient value of 9 inches for the basic pitch-pipe *Huang Chung.*[29] The underlying formula given by the prince is rather involved:

$$\textit{Huang Chung} = \frac{10^{12} \times X^{12}}{9^{11} \times 2^5} = 9, \text{ whereby } X = \frac{900.000.000}{749.153.538}.$$

$$\text{Kuttner-Kuo: } X = \sqrt[12]{\frac{9^{12}}{10^{12}} \times 2^5} = \frac{9}{10} \times 2^{\frac{5}{12}} = \frac{900.000.000}{749.153.538}.$$

(p. 22 b.)

A further pipe (= *Lü*) calculation is entitled: "81 units *Lü* according to the new method [of figuring]." These 81 units of basic measure represent, as well, the ancient 9-inch tradition wherein the inch is subdivided into 9 instead of 10 parts: 9 X 9 = 81. The explanation speaks of *tsung shu* (= vertical length) measured in "81 units *Lü*" and lists the 12 semitones in the following up-and-down "circular" order:

$$C = 8.1 \qquad G = \frac{8.1}{0.81} X \qquad D = \frac{8.1}{0.81^2} \times \frac{X^2}{2}$$

$$A = \frac{8.1}{0.81^3} \times \frac{X^3}{4} \qquad E = \frac{8.1}{0.81^4} \times \frac{X^4}{4} \; \ldots \text{etc.}$$

$$C = \frac{8.1}{0.81^{12}} \times \frac{X^{12}}{32} = 8.1$$

$$\text{Chu defines: } X = \frac{810.000.000}{749.153.538}.$$

$$\text{Kuttner-Kuo: Since } \left(\frac{X}{0.81}\right)^{12} = 32,\ X = 0.81 \times \sqrt[12]{32} = \frac{810.000.000}{749.153.538}.$$

We have here, of course, another application of the 749-tempering method, applied to the up-and-down principle of tone generation and based on the value 81 as starting point.

(p. 24 b.)

Chapter 2. Outer and Inner Diameters of the Lü Pipes

The chapter opens with a drawing demonstrating the outer and inner diameters of 36 pitch-pipes through three octaves, from C through C_1, C_2 and on to B_2. The three octave ranges are, of course, double *Lü*, single *Lü* and half *Lü*. It turns out that, in Chu's computation, the inner diameter of each individual semitone equals the outer diameter of the same semitone in the next higher octave range. Thus, the pipe C_2 fits telescopically into C_1; C_1 into C; $C\sharp_2$ fits into $C\sharp_1$; $C\sharp_1$ into $C\sharp$, etc. This is a very neat and aesthetically pleasing solution, but it is acoustically senseless; the intended pitch correction by decreasing diameters follows quite different acoustical laws.

(p. 32 a.)

The first of the pitch-pipe calculations sets out with a tube length of 2 feet and proceeds in semitonic order to the higher octave which equals 1 foot in length.

$C = 2$ [feet] $\quad C\sharp = 2 \times \frac{1.000.000.000}{1.059.463.094} = 2\,X$

$D = 2\,X^2 \quad D\sharp = 2\,X^3 \quad E = 2\,X^4 \ldots$ etc.

$C_1 = 2\,X^{12} = 1$ [foot].

Chu defines: $X = \frac{1.000.000.000}{1.059.463.094}$.

Kuttner-Kuo: Since $X^{12} = \frac{1}{2}$, $X = \sqrt[12]{\frac{1}{2}}$.

Here the value $\sqrt[12]{2} = 1{,}05946\ldots$, developed previously for string lengths, is analogically applied to the lengths of pitch-pipes which are to be tuned in equal temperament.

(p. 33 b.)

The next computation presents the outer circumferences of a set of pipes. Starting out with *Huang Chung* (= C) *Pei Lü* (= double octave) equalling 2/9 (feet) circumference, Chu develops:

$C\sharp$ double octave $= \frac{2}{9}\,X$, D double octave $= \frac{2}{9}\,X^2$,

$D\sharp$ double octave $= \frac{2}{9}\,X^3$, etc. up to C_1 (single octave) $= \frac{2}{9}\,X^{12}$.

Chu defines: $X = \frac{1.000.000.000}{1.029.302.236}$.

Kuttner-Kuo: Outer circumference C_2 (half octave), $X = \sqrt[24]{\frac{1}{2}}$

Outer circumference C_3 (quarter octave), $X = \sqrt[36]{\frac{1}{2}}$.

(p. 38 a.)

A condensed survey of a further series of calculations will suffice:

$$\text{Outer diameter C} = \sqrt{2 \times \left[\frac{\text{outer circumference of C} \times 9}{40}\right]^2}$$

Outer diameter C$\sharp$ = Outer diameter C $\times$ X, etc., up to

Outer diameter B = Outer diameter C $\times$ X^{11}, etc. up to X^{24}.

Chu defines: $X = \frac{1.000.000.000}{1.029.302.236}$.

Kuttner-Kuo: $X = \sqrt[24]{\tfrac{1}{2}} = \frac{1.000.000.000}{1.029.302.236}$.

(p. 44 a.)

Similarly, the inner diameters are gained as $\frac{1}{40}$ part of the length of the basic tube *Huang Chung*:

C inner diameter = C, total length $\times$ $\frac{1}{40}$

C$\sharp$ inner diameter = C, inner diameter $\times$ X

D inner diameter = C, inner diameter $\times$ X^2, etc., up to

C_1 inner diameter = C, inner diameter $\times$ X^{12}

C_2 inner diameter = C, inner diameter $\times$ X^{24}.

Chu defines: $X = \frac{1.000.000.000}{1.029.302.236}$.

Kuttner-Kuo: $X = \sqrt[24]{\tfrac{1}{2}} = \frac{1.000.000.000}{1.029.302.236}$.

(p. 47 a.)

It should be noted that the multiplying factors used in the last few calculations are based on an ancient Chinese approximation[30] to the value of $\pi = \frac{20}{9} \times \sqrt{2}$, or $\frac{1}{\pi} = \frac{9}{40} \times \sqrt{2}$.

The next proposition relates the inner diameter to the inner circumference of the pipes through the three octave ranges.

$\sqrt{\tfrac{1}{2}\,(\text{C inner diameter})^2} \times \frac{40}{9}$ = C inner circumference

C$\sharp$ inner circumference = C inner circumference $\times$ X, etc. up to

C_1 inner circumference = C inner circumference $\times$ X^{12}, and up to

C_2 inner circumference = C inner circumference = X^{24}.

Chu defines: $X = \frac{1.000.000.000}{1.029.302.236}$.

Kuttner-Kuo: $X = \sqrt[24]{\tfrac{1}{2}} = \frac{1.000.000.000}{1.029.302.236}$.

(p. 52 a.)

Chapter 3. Surface Area, Bore and Volume of the Pipes

The first calculation presents the inner surface area (inner bore):

$$\sqrt{\left[2 \times \left(\frac{\text{C inner circumference} \times 9}{40}\right)^2\right]^2 \times \frac{100}{162}} = \text{C inner surface area}$$

C♯ inner surface area = C inner surface area $\times$ X

D inner surface area = C inner surface area $\times$ X^2, etc., up to

C_1 inner surface area = C inner surface area $\times$ X^{12} and up to

C_2 inner surface area = C inner surface area $\times$ X^{24}.

Chu defines: $X = \frac{1.000.000.000}{1.059.463.094}$.

Kuttner-Kuo: $X = \sqrt[12]{\frac{1}{2}} = \frac{1.000.000.000}{1.059.463.094}$. $\pi = \frac{20}{9}\sqrt{2}$

(p. 58 b.)

The last proposition of chapter 3 presents the outer volumes of 36 *Lü* pipes through the three octave ranges. We use here, as a constant,

a = outer volume of *Huang Chung, Pei Lü*, i.e., C of the double (lowest) octave.

Chu defines: a = 3.928.371.006.591.930

$$x = \frac{1.000.000.000.000.000.00}{1.122.462.048.309.372.98}.$$

According to his formula, the outer volume of

$C\sharp = a \times X$ $\qquad C_1 = a \times X^{12} = \frac{a}{4}$ $\qquad C_1\sharp = a \times X^{13}$

$C_2 = a \times X^{24} = \frac{a}{16}$ $\qquad C_2\sharp = a \times X^{25}$ $\qquad B_2 = a \times X^{35}$.

Kuttner-Kuo: Since $X^{12} = \frac{1}{4}$, $X = \sqrt[12]{\frac{1}{4}} = \frac{1.000.000.000.000.000.00}{1.122.462.048.309.372.98}$.

$$a = \text{length of pipe} \times r^2\pi = 3.928 \ldots.$$

(p. 63 a.)

One cannot help reflecting on the discrepancy between the accuracy of X and a, which are given to 16 or 18 decimals, and the inaccurate value used for π in the same calculation.

Chapter 4. History of Calculation Techniques

Here the Prince gives the traditional method of the "circular" up-and-down principle, using the 2/3 and 4/3 multiplication factors. His result, of course, is the equivalent of the Western "Pythagorean" system, 12 steps of perfect fifths (3:2) and 7 octave transpositions (1:2):

$$\left(\tfrac{3}{2}\right)^{12} \times \left(\tfrac{1}{2}\right)^{7} = \frac{531.441}{524.288}.$$

Chu's version arrives, with the 11th step, at *Chung Lü* (= E♯):

$$\left(\tfrac{2}{3}\right)^{6} \times \left(\tfrac{4}{3}\right)^{5} = \frac{65.536}{177.147}.$$

If one carries this series and value one step further, from E♯ back to the starting tone C ($\times\frac{4}{3}$) and adds one more octave transposition, up to the higher C_1 ($\times\frac{2}{1}$), one finds the 13th tone of the complete system and the above fraction which defines, in Western tradition, the ditonic comma.

The rest of Chapter 4 deals with a comparison of the "old and new methods" of calculating the *Lüs*; the new method is, of course, Prince Chu's equal temperament solution.

Chapter 5, the remaining section of the *Lü Lü Ching I,* is essentially historical in character and gives four references for the "old methods," among them *Huai Nan Tzu* (c. 122 B.C.) and *Ching Fang* (c. 45 B.C.). There are no further original computations by Prince Chu in this final chapter which deals, among other subjects, with the four traditional standard measures for the length of the basic pipe *Huang Chung* (= C): 10 inches, 9 inches, 8.1 inches, and 9 inches subdivided into 9 parts instead of 10 decimal parts.

The final pages discuss the technology of making pipes, and Chu actually recommends a certain type of bamboo (sic!) from the Chin Men mountain. This is followed by advice as to how the pipes should be played (i.e., intoned), and how a comparison between the "old and new methods" can be made by testing their sounds (i.e., pitches). The section ends with a reflection on the qualities of musical sounds, quoting liberally from historical sources.[31]

Concerning the calculations in Chapters 2 and 3 which deal with the circular dimensions of the pitch-pipes (outer and inner diameters, circumferences, surface areas and total volumes), the following should be stated. Reaching back into the early centuries of our era, a lively argument went on between the theorists of the *Lü* as to whether the lengths of the pipes could, or could not, be rigidly set by the principle of the 2/3 division. One faction held that the inner diameters of the pipes must be reduced from one semitone to the next, or else, that the pipe lengths must be slightly modified from semitone to semitone, in deviation from a strict 2/3 division, if the inner diameters are not increasingly reduced throughout the whole set. The other side advocated precise adherence to the 2/3 division without adjustment of inner diameters. From the theoretical viewpoint of pipe acoustics obviously the first argument was right, the second wrong. And in so far as Prince Chu reasoned correctly in his search for a formula which should guide the modification of all circular dimensions of the pitch-pipes. This is apparently what Robinson (1962:186, note d; 213) had in mind when he contended that Chu had discovered the "physics of end effect" because he "compensated [for the same] by also tempering their diameters, dividing each successive diameter by the 24th root of 2" (Robinson 1962: 224).

Now, the "acoustics of end effect" were at least in part already known in Chinese antiquity, as evidenced by the arguments mentioned above. Thus, the discovery of these effects cannot be credited to Prince Chu. So much about the theoretical aspects of this problem. But when we turn to the practice of pipe intonation and pitches, all of Chu's pipe calculations collapse as playful speculations, invalidating Robinson's praise of them. The first irrational proposition is that of dimensions and the possibility of their technological manipulation. The longest tube proposed by Chu measures 2 feet for C double octave, which corresponds to 1 foot for C single octave, or 6 inches for C half octave. The inner diameter is to be $\frac{1}{40}$ of that length = $\frac{6}{40}$ in. = 3.8 mm. The next lower C should have the diameter $\frac{12}{40}$ in. = 7.62 mm. The difference between these two has to be divided by some value near 12 in order to approximate the diameter adjustment from pipe to pipe in that octave range. This amounts to roughly 3/10 of a millimeter or close to 1/100 of one inch. There was, of course, no technology available in the China of 1595/96 to machine the bore of 36 metal tubes precisely to such narrow tolerances, nor did the Chinese have machine tools for manufacturing drills to such stringent specifications. Moreover, precision pitch-pipes during the Ming period were made of jade, an extremely hard and brittle material which would have broken in the attempt to cut and drill tubes of these dimensions.

It follows that Prince Chu could not have made a set of 36 pitch-pipes to fit his own numerical specifications and to test their acoustical properties. That leaves most of his calculations in chapters 2 and 3 in the area of theoretical speculation, apart from the misinterpretation of pipe acoustics in general.

A second, equally serious objection concerns the unreliability of pipe intonation. Embouchure, lip pressure at the blowhole, air pressure, angle of pipe position against the player's mouth can change intonation and pitch easily up to a full semitone, a fact of which every woodwind player in a modern symphony orchestra is aware. These conditions are severely aggravated when dealing with primitive pipe shapes, and when the lengths of the sonant tubes are as small as in the case of Chu's propositions.

As early as c. 45 B.C. did the Chinese diviner and acoustician Ching Fang realize that pipes are unsuitable for precise acoustical measurements and that one has to use strings in order to arrive at accurate results.[12] In modern times Bukofzer demonstrated the same fact when he used mechanical-pneumatic means to incite pitch-pipes and failed to achieve results reliable enough to support von Hornbostel's *Blasquintentheorie*. As a consequence many ethnomusicologists believe this theory to have been disproved by Bukofzer's experimental series (Bukofzer 1936).

All these considerations force us to conclude that Prince Chu's calculations for pitch-pipes tuned to equal temperament are worthless, in spite of the enormous effort and ingenuity evident in the respective two chapters:

1—the acoustical premises underlying his formulas are wrong;
2—the mathematical approach is inadequate for precision results in various instances because of the inaccurate value used for π;
3—experimental proof for his theoretical findings is impossible due to the non-existence of a technology that could meet exorbitant tolerance requirements;
4—his whole concept of the problem was unworkable: to design primitive and unreliable instruments for the production of highly precise pitches, as are required for the minute deviations from "pure" intonation which form the essence of equal temperament.

A totally different situation exists for the computation of string lengths in chapter 1. Here perfectly accurate results are presented which testify to the Prince's mathematical competence and reasoning power. Not the slightest doubt can be raised as to the theoretical value and correctness of Chu's presentation of equal temperament in monochord form. That his nation had no practical use for, and no interest in, his solution is unfortunate but not unique. Many inventors and formulators throughout history saw their achievements fall by the wayside without reward and recognition.

APPENDIX III

Prince Chu's Own Comments on the Priority Question

Chu's attitude concerning the originality of his equal-temperament solution is, at least in part, strangely ambivalent. On the one hand he names a variety of ancient sources, such as the *Huai-nan Tzu*, the *Chin Shu* and *Sung Shu* as his procedural references or predecessors. He even goes as far as superimposing the 749-method on the *Huai-nan Tzu* monochord with which it has nothing whatever to do. He quotes the *Chin Shu* where the divisor 749 originated, as a reference, but without mentioning that crucial number. Thus, one gets the impression that he tries to enter a priority claim for the 749-procedure, but wants to be on the safe side in case he should be suspected of plagiarizing from the *Chin Shu*: after all, hasn't he listed that source as a reference? Conceivably the mystification with the *Huai-nan Tzu* could have been a trick to mislead the reader and steer him away from the real source, i.e., the *Chin Shu.*

On the other hand, he states: "I have founded a new system" (No. 4, chapter 1, p. 5b), and in his later life he emphasized that he considered his equal-temperament solution as his most important achievement in which he took particular pride. This vacilation between claiming and not quite claiming original authorship is very difficult to explain, unless one assumes that he wanted to protect himself against criticism from traditionalists.

There are two pertinent quotations from the 1584 work which seem to contribute to an interpretation of the priority situation; both are introduced by Robinson for his misguided attempt to prove Prince Chu's extraction and use of the twelfth root of 2.

> I had made an attempt with the theory of the *Sung* [scholar] *Chu Hsi,* based on the ancient up-and-down principle, and using this tried to get the positions for the standard pitches on the zither. But I noticed that the [normal] notes of the zither were not in consonance with [those produced from] the positions of the standard pitches, and suspicions therefore arose in my mind.
>
> Night and day I searched for a solution and studied exhaustively this pattern-principle. Suddenly early one morning I reached a perfect understanding of it and realised for the first time that the four ancient sorts of standard pitches all gave mere approximations to the notes. This moreover was something which pitch-pipe exponents had not been conscious of for a period of two thousand years.
>
> Only the makers of the zither [*ch'in*] in their method of placing the markers at three-quarters or two-thirds [etc. of the length of the strings] had as common artisans transmitted by word of mouth [the way of making the instrument] from an unknown source. I think that probably the men of old handed down the system in this way, only it is not recorded in literary works (No. 4, chapter 1, 5a; trans. Robinson 1962:221).

From these sentences Robinson concludes: ". . . one might infer that Chu Tsai-Yü had recovered the secret of equal temperament from the remotest antiquity, but in fact he does not say so." Clearly, Robinson as well is puzzled and even misled by Chu's ambivalence. Actually, this quotation shows only two things:

1–Chu had realized that there are considerable differences between the pitches of "Pythagorean" intonation governing the traditional Chinese pitch-pipe system, and those of the seven-stringed zither *ch'in* which had, since antiquity, always been tuned to just intonation pitches;
2–In 1584 the Prince was still fairly ignorant of some basic facts of his nation's two tuning systems and their differences. Otherwise, he

could not have been so surprised and pleased upon discovering that the just *ch'in* pitches were "mere approximations" to the frequencies of the up-and-down principle.

The second quotation from the *Lü Hsüeh Hsin Shuo* reads as follows:

> I have founded a new system. I establish one foot as the number from which the others are to be *extracted,* and using [*square and cube root*] *proportions,* I extract them. Altogether one has to find the exact figures for the pitch-pipes in twelve *operations* (Italics added) (No. 4, chapter 1, 5b; trans. Robinson 1962:223).

This translation must be rejected as a severely distorting misinterpretation. The term "extract" (which applies specifically to root-extracting operations) should have been given as "calculate." The four words in brackets ["square and cube root"] are not contained in the original and represent an arbitrary interpolation; they create the wrong impression that here the square and cube root extractions are being referred to as they would occur typically in equal temperament calculations based on the twelfth root of 2. Furthermore, the words "proportions" and "operations" do not occur in the text; the original speaks only of subtractions and divisions. If we remove these liberties from the translation, nothing remains to establish a connection to root-extracting operations.

What this quotation actually refers to is the sequence of ordinary divisions which make up the complete 749-procedure described above in Appendix I.

By now sufficient evidence has been examined to determine what precisely had been achieved by Chu in both of his treatises. The 1584 work aimed at, and solved, the elimination of the ditonic comma by applying a reduced size of fifth and fourth, i.e., by tempering. This solution did away with the spiral effect of the traditional tone system and produced a closed circular or chromatic system with a perfect octave. That these monochords created, as a by-product, 12 equally-sized semitones, i.e., equal temperament, was an unexpected coincidence. It is highly unlikely that the Prince was aware of that important by-product; in fact, all evidence speaks against it, especially because the procedure of repeated manipulation by the 749-divisor does not suggest and facilitate the discovery of that hidden result.

By the time the *Lü Lü Ching I* was completed eleven years later, Chu had thoroughly investigated his earlier monochords, discovered four additional divisors and various hidden relationships between the individual semitones. We can thus be sure that in 1595/96 Chu knew he was dealing with equally-sized intervals, hence with equal temperament, irrespective of the fact whether by then he may have discovered the possible role of the twelfth root of 2.

In the preface of the second work Chu makes another claim that he has discovered a new system never described anywhere before. If he had really "discovered" equal temperament already in 1584, it would be impossible to explain what this second discovery was. Whether the Prince realized that the pitch numbers of his monochords represented eleven mean proportionals between the extremes of the octave ratios, is both uncertain and irrelevant because that would be just one form of possible mathematical definition. But there is little doubt that by 1595 he had recognized the equal-size aspect of his semitones.

By forcing the issue of the "second discovery" it would be possible to argue that Chu's system of equal-temperament calculation was not completed until 1595/96, thus invalidating the priority date of 1584. Under no circumstances would we support this kind of argument, because the 1584 monochords offered perfect and complete equal-temperament figures. Whether or not the Prince immediately recognized all or only part of the qualities of his results, is immaterial: the date of an invention or discovery does not depend on how much insight the originator had into all its properties or consequences at a given time.

SUMMARY

Chu Tsai-yü presented a highly precise, simple and ingenious method for arithmetic calculation of equal-temperament monochords in 1584. Stevin offered a mathematical definition of equal temperament plus a somewhat less precise computation of the corresponding numerical values in 1585 or later. That clearly establishes a priority for Prince Chu by at least one year, possibly by quite a few more years. Both scholars worked in complete independence and without any knowledge of each other's work; although their final numerical results were in part nearly identical, their approaches and methods were fundamentally different and unrelated.

It seems pointless to attempt a qualitative evaluation of the two solutions with a view to label one or the other as superior. Both authors proceeded within the framework of their own cultural traditions and created perfect solutions consistent with their different scientific surroundings. Neither Chu nor Stevin can be recognized as "inventors" of equal temperament, for the following reasons. In China the problem of such tempered tuning had been known for as many centuries as in Europe. Stevin's definition and calculation was not needed in Europe where practical tunings with satisfactory approximations to good tempering had been known and used for almost a hundred years before his work. In Chinese musical practice equal temperament was neither needed nor desired before or after Chu's publications, thus unfortunately making his achievement irrelevant for his nation.

Both solutions were computing procedures, with the computing priority going to Prince Chu. Furthermore, they were algebraic or arithmetic procedures more related to applied mathematics than to musical acoustics. This invalidates the opinion that Chu's contribution was "the crowning achievement of China's two millennia of acoustical experiment and research."

It has not been the aim of the present paper to question or belittle Prince Chu's work, but rather to adjust exaggerated claims made in his behalf by other authors to a more realistic appraisal of this great scholar's theoretical work. He has amply deserved a place of honor in the history of musical temperament and his position remains strong in spite of some of the reasoning in these pages. It is our hope that his solution will soon become widely known as "Prince Chu's 749-Temperament."

One important task remains for Western and Chinese musicology alike. The removal of the aura of glamour or sensation from Prince Chu's temperament contribution must now lead to a full recognition of his writings on musical history which have been inexcusably neglected and underrated, overshadowed, as they were, by the undue emphasis on the *Lü Hsüeh Hsin Shuo* and the *Lü Lü Ching I.* It is our conviction that many of the titles listed above will establish Chu Tsai-yü eventually as one of the greatest historians of his nation's music.

NOTES

1. For the transliteration of Chinese names and terms we are using the English Wade-Giles system.

2. This combination of disciplines may seem odd today, but it was the traditional one in China since at least the third century B.C. It has its parallels during the late Renaissance period in Europe when a unity of the sciences was still possible and, in fact, postulated. The close connection between the calendar and the theory of music, however, has few precedents in the Occident.

3. I wish to acknowledge my indebtedness to the Ming Dynasty Biographical History Project at Columbia University, New York, under the direction of Professor L. Carrington Goodrich. Having been granted access to the files of the Project, valuable information about some life data and works of Chu Tsai-yü could be incorporated into the corresponding sections of this paper. According to Professor Goodrich the first volume of the Ming Dynasty Biographical Dictionary, with a lengthy article on Prince Chu, is scheduled for publication in May 1975.

4. A tenth digit is occasionally quoted in Western literature because of an error committed by Père Jean Joseph Marie Amiot (Mémoire sur la Musique des Chinois tant anciens que modernes, Paris 1780). In this work (Fig. 18, planche XXI), Amiot gives a listing of Chu's complete monochord which contains a tenth digit throughout: each of the 12 semitone values has an additional zero in the sixth decimal place which does not belong there. Incidentally, Amiot does not name Prince Chu as his source; he just states that the calculations have been done "par les Chinois modernes."

5. The twelfth root of 2 is the classical Western definition of a semitone in equal temperament, and the 12 equally tempered semitones of the monochord are defined, since Simon Stevin, as the 12 consecutive powers of $\sqrt[12]{2}$. The reasons why this mathematical expression is never stated in Chu's three treatises dealing with equal temperament is examined in Appendix No. I.

6. It is with regret that these pages bring me into conflict with some of Robinson's views. I hasten to emphasize that I consider his essay in Needham's volume as one of the most important and valuable Western contributions to the history of Chinese musical acoustics since Maurice Courant's work in the Lavignac Encyclopédie, Paris 1912.

My dissension, then, concerns only the pages 212-28 which deal with the development of temperaments east and west, and with the priority problem itself.

7. Stevin (1585:52) wrote "Problème XLV: Etant donnez deux nombres quelquonques: Trouvez leurs moyens proportionels requis." "Example I: *Explication du donnè*. Soyent donnez deux nombres 2 & 10. *Explication du requis.* Il faut trouver leur moyen proportionel. *Construction.* On Multipliera 2 par 10, font 20, des mesmes la racine quarrée est $\sqrt{20}$. Je di que M20 est le moyen proportionel requis." (See Figure No. II).

8. Fokker (1966, Vol. V) does not offer any further clues on the dating question. In a personal communication to me (July 4, 1968) Prof. Fokker merely states that Stevin's *Arithmètique* was published in 1585, and that this should be the year for the mathematical solution of constructing two and more mean proportionals and extracting highpower roots. He also writes that "Stevin ist nicht der Erfinder [of equal temperament], er ist der Errechner,"–a reasonable statement with which one may agree. Fokker's edition is valuable to interested readers because it gives a complete English translation side by side with the original Dutch text.

9. One proportional: multiply the two extremes, extract the square root from the product. Two proportionals: take the product and extract the square root for the first, then the cubic root for the second proportional. Three proportionals: extract from the product the square, cubic and quartic roots. The same procedure applies to four and five mean proportionals which require, in addition to the above, quintic and sixth roots. It follows by analogy that for eleven mean proportionals the square through twelfth roots have to be extracted from the product. Since in the case of the 12 semitones the two given numbers are 1 and 2 (standing for the higher and lower octave), their product is $1 \times 2 = 2$, and the twelfth root of 2 will produce the first of the 12 semitones. As concerns the roots, only a cubic root gives trouble; quartic roots can be extracted by computing twice a square root, and a twelfth root by extracting first a cubic root and then twice the square roots from the result. Finally, if $\sqrt[12]{2} = i$, the powers of i (i^2, i^3, i^4 etc. up to i^{12}) will produce the other ten semitone values. In this way 7 of the 12 monochord figures can be gained by simple multiplication, once the twelfth root has been extracted.

10. This work was not written by Liu An himself, but by a group of scholars assembled at his court and supported by him.

11. A passage of Boethius, *De Institutione Musica,* III, 8, suggests the possibility that Philolaus (second half of 5th century), a disciple of the Pythagorean school, may have constructed a 53-division of the octave. Cf. Barbour 1951:123.

12. *Hou Han Shu* (= History of the Later Han Dynasty, 25-220 A.D.), *Lü Li Chih* (= Treatise of the *Lü* [= Music Theory] and the Calendar), p. 4a: "Again, according to Ching Fang, the sound of bamboo [pipes] cannot be tuned precisely. Therefore, he made an instument called *chun* to determine the figures. The shape of the *chun* was like a *sê* (an ancient multi-stringed zither); the length of it was one *chang* (equal to ten Chinese feet = 141 inches of modern measure). According to the [theory of the] *Lü*, [the length of] *Huang Chung* (= C) was 9 inches. Therefore he made the space between the bridges [i.e. the total string length] 9 feet. Under the central string he incised decimal parts of inches to provide subdivisions for measuring the "clear" and the "muddy" sound [i.e., the highest and lowest of the traditional three octaves] of the 60 Lü." (Translation contributed by Prof. Larry N. Shyu). The *chun* had 13 strings for the 12 semitones plus the higher octave C. Chu also used a multi-stringed measuring instrument of the *chun* type to test the acoustical results of his temperament figures. A drawing shows this instrument to have 12 strings only, and the explanations going with the figure confirm this number. One wonders why he did not use 13 strings, because with his *chun* he could not test too well the tone B, or any other semitone for that matter, against the higher octave C_1.

13. Barbour has shown (1951:56), how Ho Ch'êng-t'ien's monochord could have been greatly improved if he had started his calculations with the lower octave, working

upwards, instead of working down from the higher octave. This would have reduced the mean deviation from 4.9 to 2.2 cents.

14. See also Barbour 1951:50 ff., where a variety of mechanical and geometrical methods for equal temperament constructions are discussed, especially the three methods given by Gioseffo Zarlino in 1588.

15. Traditionally and consistently, Chinese theorists have always ended their calculations with step no. 11, E♯ = $\frac{177.147}{131.072}$. Since the multiples of 2 in the sequence 2:3 × 4:3 × 2:3 . . . etc. seem to be irrelevant (as mere indicators of octave position), they called the value 177.147 ($=3^{11}$) "the big number" of the tone system. Obviously, the ancient Chinese knew very well that the twelfth step to B♯ does not produce a useful interval, i.e., a perfect octave. This makes it highly probable that, already in early antiquity, they preferred a perfectly tuned octave, ratio 2:1, and had no use for the comma-distorted twelfth pitch.

16. John Napier's first logarithmic table was published in 1614. Henry Brigg's improved tables appeared in 1624 and contained 14-place common logarithms from 1-20.000 and 90.000-100.000.

17. The official history of the Chin Dynasty (265-419 A.D.) completed 635 during the T'ang Dynasty.

18. The official history of the [Liu] Sung Dynasty (420-478 A.D.), completed 500 during the Southern Ch'i Dynasty.

19. Reinhard (1956:79-80) correctly recognized the 750:749 temperament, but he failed to account in his interval tabulation for the "magical remainders" left over after each division. As a consequence, the cents values given in his table represent perfect fifths (divided by 750) for the first four steps instead of tempered fifths (divided by 749); the remaining eight steps are also misstated, owing to similar errors.

20. Quoted from *Lü Lü Ching I* (1595/96), ch. 4, pp. 84a, 88a, 89b.

21. *Chin Shu,* treatise no. 6 on *Lü* [and] *Li*, i.e., the semitones and the calendar.

22. *Chiu Chang Suan Shu* = Nine Chapters on the Mathematical Art. Later Han Dynasty (first century), containing much material from the Former Han and perhaps Ch'in Dynasty. Writer unknown.

23. Such as *Chu Shih-chieh, Suan Hsüeh Ch'i Mêng* = Introduction to Mathematical Studies, 1299, and *Ssu Yuan Yü Chien* = Precious Mirror of the Four Elements [of algebra], 1303, both Yuan Dynasty.

24. I gratefully acknowledge the assistance of Mr. Jacob Sturm, mathematics student at Columbia College, New York City, in speculating about methods possibly used by Chu for the determination of the additional six decimals and for root extraction.

25. These are: the nine-digit monochord in circular progression with octave ratio 10:5, the five-digit 36-pitch monochord through three octaves, ratios 200:100:50:25 in chromatic order, and another one with ratio 81:40.5.

26. Because of its significance for the priority question Robinson's report in Needham's volume IV deals almost exclusively with the earlier 1584 work and mentions the *Lü Lü Ching I* only with a few words.

27. I gratefully acknowledge the assistance of Dr. Kuo Chung-wu, Mathematics Ph.D., formerly at Columbia University, who helped in securing correct interpretations of all mathematical aspects of the *Lü Lü Ching I.* Dr. Odoric Y. K. Wou, Assistant Professor of History, Rutgers University, assisted in the translation of texts from Prince Chu's works, for which I express my gratitude and indebtedness.

28. The traditional three Chinese octave ranges are called *Pei Lü* (= double *Lü*, lowest), *Chêng Lü* (= single *Lü*, middle) and *Pan Lü* (= half *Lü*, highest).

29. The text says: "Slanting length (*hsieh shu*) 90[units] *Lü* according to the *new method.*" The new method, of course, is the 749-procedure. And the 90 units mean that each of the 9 inches is subdivided into 10 decimal parts.

30. This is not a good approximation. In antiquity, the Bible gives a value for π as 3. The same value appears in the arithmetics of Han time China (202 B.C.-220 A.D.). Archimedes, using the method of exhaustion and a 96-sided polygon, computed (c. 250

B.C.) an approximation to π as $\frac{22}{7}$ = 3.1428. On the same level is the formula $\frac{20}{9} \times \sqrt{2}$ = 3.1427. But already around 480 A.D. the Chinese mathematician Tsu Ch'ung-Chih (c. 430-501?A.D.) developed the value $\frac{355}{113}$ = 3.141592+, which is precise to 6 decimals. Prince Chu found the formula $\frac{9}{40} \times \sqrt{2}$ convenient for these procedures and could use it in spite of its inaccuracy because he worked here with ratios relating one semitone to the next where π is being cancelled out in some, but not in all the calculations.

31. This ends Book I of the *Lü Lü Ching I.* Book II, with chapters 6-10, deals further with historical developments including such topics as ancient weights and measurements, musical instruments of antiquity, etc. The contents of these chapters, then, has no bearing on the theory of equal temperament.

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LES

ŒUVRES Mathematiques

DE

SIMON STEVIN de Bruges.

Ou ſont inſerées les

MEMOIRES MATHEMATIQVES,

Eſquelles s'eſt exercé le Tres-haut & Tres-illuſtre Prince MAURICE de NASSAU, Prince d'Aurenge, Gouverneur des Provinces des Païs-bas unis, General par Mer & par Terre, &c.

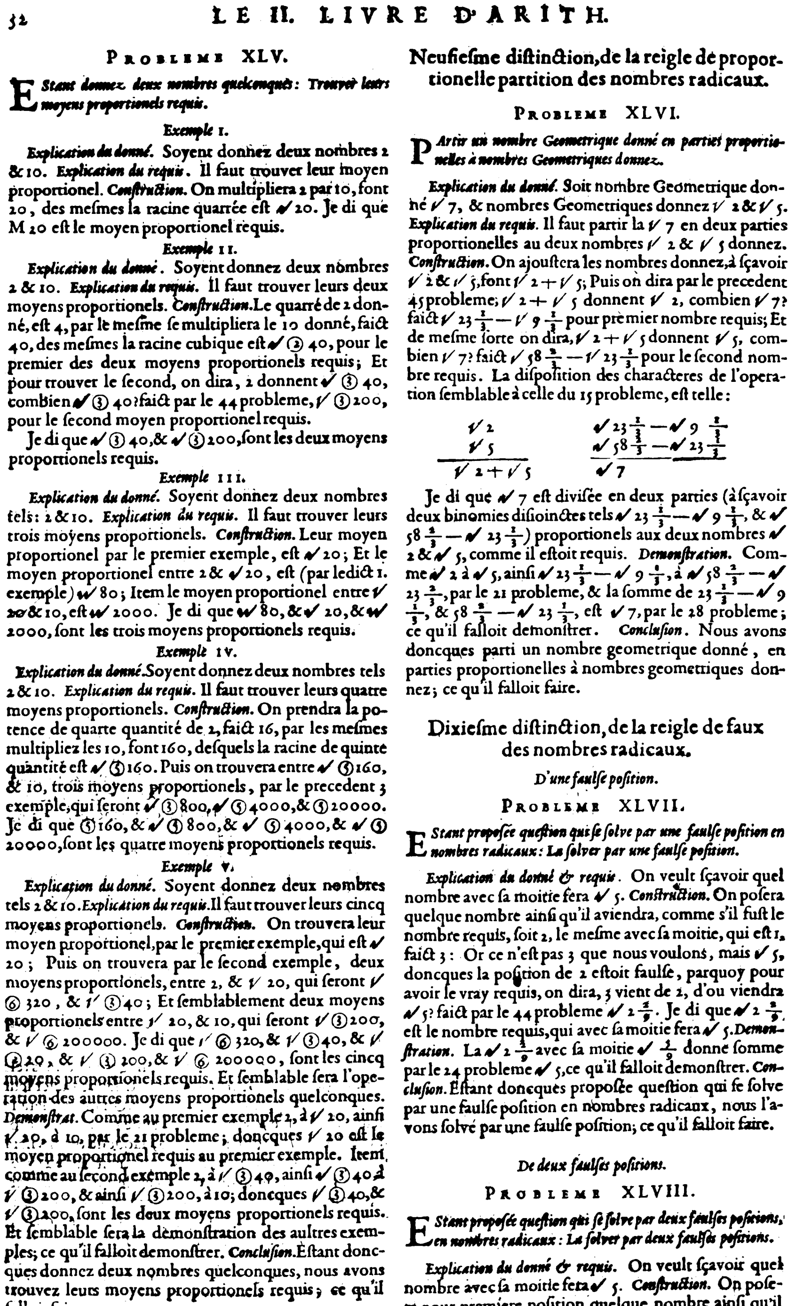
Le tout reveu, corrigé, & augmenté,

Par ALBERT GIRARD Samielois, Mathematicien.

A LEYDE

Chez Bonaventure & Abraham Elſevier, Imprimeurs ordinaires de l'Univerſité, ANNO CIↃ IↃ C XXXIV.

Figure I. Title Page of Stevin's "Arithmétique" from the second edition 1634.



32 **LE II. LIVRE D'ARITH.**

PROBLEME XLV.

Estant donnez deux nombres quelconques: Trouver leurs moyens proportionels requis.

Exemple I.

Explication du donné. Soyent donnez deux nombres 2 & 10. *Explication du requis.* Il faut trouver leur moyen proportionel. *Construction.* On multipliera 2 par 10, font 20, des mesmes la racine quarrée est $\sqrt{20}$. Je di que M 20 est le moyen proportionel requis.

Exemple II.

Explication du donné. Soyent donnez deux nombres 2 & 10. *Explication du requis.* Il faut trouver leurs deux moyens proportionels. *Construction.* Le quarré de 2 donné, est 4, par le mesme se multipliera le 10 donné, faict 40, des mesmes la racine cubique est $\sqrt{③\,40}$, pour le premier des deux moyens proportionels requis; Et pour trouver le second, on dira, 2 donnent $\sqrt{③\,40}$, combien $\sqrt{③\,40}$? faict par le 44 probleme, $\sqrt{③\,200}$, pour le second moyen proportionel requis.

Je di que $\sqrt{③\,40}$, & $\sqrt{③\,200}$, sont les deux moyens proportionels requis.

Exemple III.

Explication du donné. Soyent donnez deux nombres tels: 2 & 10. *Explication du requis.* Il faut trouver leurs trois moyens proportionels. *Construction.* Leur moyen proportionel par le premier exemple, est $\sqrt{20}$; Et le moyen proportionel entre 2 & $\sqrt{20}$, est (par ledict 1. exemple) $\sqrt{\sqrt{80}}$; Item le moyen proportionel entre $\sqrt{20}$ & 10, est $\sqrt{\sqrt{2000}}$. Je di que $\sqrt{\sqrt{80}}$, & $\sqrt{20}$, & $\sqrt{\sqrt{2000}}$, sont les trois moyens proportionels requis.

Exemple IV.

Explication du donné. Soyent donnez deux nombres tels 2 & 10. *Explication du requis.* Il faut trouver leurs quatre moyens proportionels. *Construction.* On prendra la potence de quarte quantité de 2, faict 16, par les mesmes multipliez les 10, font 160, desquels la racine de quinte quantité est $\sqrt{⑤\,160}$. Puis on trouvera entre $\sqrt{⑤\,160}$, & 10, trois moyens proportionels, par le precedent 3 exemple, qui seront $\sqrt{⑤\,800}$, $\sqrt{⑤\,4000}$, & ⑤ 20000. Je di que ⑤ 160, & $\sqrt{⑤\,800}$, & $\sqrt{⑤\,4000}$, & $\sqrt{⑤\,20000}$, sont les quatre moyens proportionels requis.

Exemple V.

Explication du donné. Soyent donnez deux nombres tels 2 & 10. *Explication du requis.* Il faut trouver leurs cincq moyens proportionels. *Construction.* On trouvera leur moyen proportionel, par le premier exemple, qui est $\sqrt{20}$; Puis on trouvera par le second exemple, deux moyens proportionels, entre 2, & $\sqrt{20}$, qui seront $\sqrt{⑥\,320}$, & $\sqrt{③\,40}$; Et semblablement deux moyens proportionels entre $\sqrt{20}$, & 10, qui seront $\sqrt{③\,200}$, & $\sqrt{⑥\,200000}$. Je di que $\sqrt{⑥\,320}$, & $\sqrt{③\,40}$, & $\sqrt{20}$, & $\sqrt{③\,200}$, & $\sqrt{⑥\,200000}$, sont les cincq moyens proportionels requis. Et semblable sera l'operation des autres moyens proportionels quelconques. *Demonstrat.* Comme au premier exemple 2, à $\sqrt{20}$, ainsi $\sqrt{20}$, à 10, par le 21 probleme; doncques $\sqrt{20}$ est le moyen proportionel requis au premier exemple. Item, comme au second exemple 2, à $\sqrt{③\,40}$, ainsi $\sqrt{③\,40}$ à $\sqrt{③\,200}$, & ainsi $\sqrt{③\,200}$, à 10; doncques $\sqrt{③\,40}$, & $\sqrt{③\,200}$, sont les deux moyens proportionels requis. Et semblable sera la demonstration des aultres exemples; ce qu'il falloit demonstrer. *Conclusion.* Estant doncques donnez deux nombres quelconques, nous avons trouvez leurs moyens proportionels requis; ce qu'il falloit faire.

Neufiesme distinction, de la reigle de proportionelle partition des nombres radicaux.

PROBLEME XLVI.

Partir un nombre Geometrique donné en parties proportionelles à nombres Geometriques donnez.

Explication du donné. Soit nombre Geometrique donné $\sqrt{7}$, & nombres Geometriques donnez $\sqrt{2}$ & $\sqrt{5}$. *Explication du requis.* Il faut partir la $\sqrt{7}$ en deux parties proportionelles au deux nombres $\sqrt{2}$ & $\sqrt{5}$ donnez. *Construction.* On ajoustera les nombres donnez, à sçavoir $\sqrt{2}$ & $\sqrt{5}$, font $\sqrt{2}+\sqrt{5}$; Puis on dira par le precedent 45 probleme; $\sqrt{2}+\sqrt{5}$ donnent $\sqrt{2}$, combien $\sqrt{7}$? faict $\sqrt{23\frac{1}{3}} - \sqrt{9\frac{1}{3}}$ pour premier nombre requis; Et de mesme sorte on dira, $\sqrt{2}+\sqrt{5}$ donnent $\sqrt{5}$, combien $\sqrt{7}$? faict $\sqrt{58\frac{2}{3}} - \sqrt{23\frac{1}{3}}$ pour le second nombre requis. La disposition des characteres de l'operation semblable à celle du 15 probleme, est telle:

$$\begin{array}{cc} \sqrt{2} & \sqrt{23\frac{1}{3}} - \sqrt{9\frac{1}{3}} \\ \sqrt{5} & \sqrt{58\frac{2}{3}} - \sqrt{23\frac{1}{3}} \\ \hline \sqrt{2}+\sqrt{5} & \sqrt{7} \end{array}$$

Je di que $\sqrt{7}$ est divisée en deux parties (à sçavoir deux binomies disioinctes tels $\sqrt{23\frac{1}{3}} - \sqrt{9\frac{1}{3}}$, & $\sqrt{58\frac{2}{3}} - \sqrt{23\frac{1}{3}}$) proportionels aux deux nombres $\sqrt{2}$ & $\sqrt{5}$, comme il estoit requis. *Demonstration.* Comme $\sqrt{2}$ à $\sqrt{5}$, ainsi $\sqrt{23\frac{1}{3}} - \sqrt{9\frac{1}{3}}$, à $\sqrt{58\frac{2}{3}} - \sqrt{23\frac{2}{3}}$, par le 21 probleme, & la somme de $23\frac{1}{3} - \sqrt{9\frac{1}{3}}$, & $58\frac{2}{3} - \sqrt{23\frac{1}{3}}$, est $\sqrt{7}$, par le 28 probleme; ce qu'il falloit demonstrer. *Conclusion.* Nous avons doncques parti un nombre geometrique donné, en parties proportionelles à nombres geometriques donnez; ce qu'il falloit faire.

Dixiesme distinction, de la reigle de faux des nombres radicaux.

D'une faulse position.

PROBLEME XLVII.

Estant proposée question qui se solve par une faulse position en nombres radicaux: La solver par une faulse position.

Explication du donné & requis. On veult sçavoir quel nombre avec sa moitie fera $\sqrt{5}$. *Construction.* On posera quelque nombre ainsi qu'il aviendra, comme s'il fust le nombre requis, soit 2, le mesme avec sa moitie, qui est 1, faict 3: Or ce n'est pas 3 que nous voulons, mais $\sqrt{5}$, doncques la position de 2 estoit faulse, parquoy pour avoir le vray requis, on dira, 3 vient de 2, d'ou viendra $\sqrt{5}$? faict par le 44 probleme $\sqrt{2\frac{2}{9}}$. Je di que $\sqrt{2\frac{2}{9}}$, est le nombre requis, qui avec sa moitie fera $\sqrt{5}$. *Demonstration.* La $\sqrt{2\frac{2}{9}}$ avec sa moitie $\sqrt{\frac{5}{9}}$ donne somme par le 24 probleme $\sqrt{5}$, ce qu'il falloit demonstrer. *Conclusion.* Estant doncques proposée question qui se solve par une faulse position en nombres radicaux, nous l'avons solvé par une faulse position; ce qu'il falloit faire.

De deux faulses positions.

PROBLEME XLVIII.

Estant proposée question qui se solve par deux faulses positions, en nombres radicaux: La solver par deux faulses positions.

Explication du donné & requis. On veult sçavoir quel nombre avec sa moitie fera $\sqrt{5}$. *Construction.* On posera pour premiere position quelque nombre ainsi qu'il avien-

Figure II. Stevin's Proposition no. 45 Concerning Mean Proportionals. From the second edition 1634.